# Book Review

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## Analytic Combinatorics for Multiple Object Tracking

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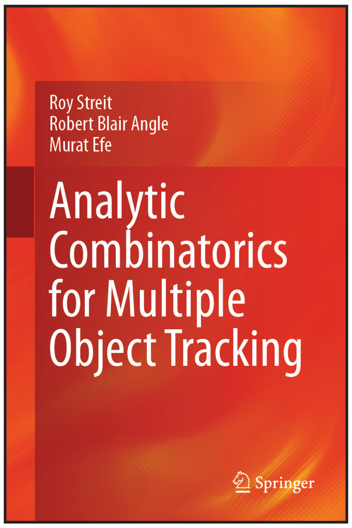
Springer, 2021
ISBN: 978-3-030-61191-0

## INTRODUCTION

Analytic combinatorics (AC), probability-generating functions (PGFs), and probability-generating functionals (PGFLs) provide a universal, exact, and intuitive approach for the structure, design, and derivation of multitarget tracking filters. This book is a must read for anyone who wants to learn the following:

- The mechanics of Bayesian tracking filters
- How existing tracking filters differ and what they have in common
- How to intuitively design a customized tracking filter
- How to derive the exact a posteriori filter equations
- How to effectively implement tracking filters based on AC and PGFLs




In particular, a closer look should be taken by engineers who want an overview of the broad field of multiobject target tracking or engineers who are often confronted with nonstandard tracking challenges and need to find customized new filter solutions to their tracking problem. It provides everything needed to understand how existing tracking filters can be extended or new filters can easily be designed. The appendix and the references provide all mathematical details to understand this innovative approach for formulating tracking filters.

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Combinatorics is a broad field of classical mathematics that comprises problems from diverse applications, ranging from theoretical considerations to highly relevant challenges in applications where, for example, objects have to be counted or enumerated, such as the measurement-to-target assignments in multiobject target tracking. Indeed, one of the main issues confronting a tracking engineer in daily work is the data association problem: which measurement (or even multiple measurements) belongs to which track? Depending on the number of measurements, the way a target generates measurements (single source [1], extended target [2], and generalized measurement model [3]), the statistics, the properties and output of the sensor(s) used, the time evolution of the target(s), the environment, etc., an immense amount of different tracking problems arise. This steady increase of challenges in multitarget tracking implied the derivation of an enormous number of different multiobject target tracking filters in past decades. Consequently, even experienced experts might find it difficult to stay on top of the developments and to distinguish benefits and drawbacks of the different approaches. The diversity of tracking filters enables a tracking engineer to solve problems in state estimation and tracking with a great level of detailed modeling. In addition, AC for multitarget tracking provides a toolbox of concepts that can be combined, extended, and generalized to solve new tracking tasks or to improve information extraction for given data. This directly implies that it is often unknown to a tracking engineer whether a solution to a new tracking scenario exists or whether related concepts can be used to derive a tailormade solution to the problem. An overview of the range of tracking approaches is hard to attain because most tracking filter derivations differ in their approach and a unified framework of representation is missing.

The authors provide the unification of such multiobject tracking filters by applying the theory of point processes, generating functions and functionals that encode the measurement-to-target data associations for several assumptions. Furthermore, the authors explain different approaches to decode the information via differentiation, including highly relevant methods for implementation of tracking filters via particle filtering. A unification of different tracking approaches thereby not only helps in understanding differences and similarities in existing approaches but is also a valuable contribution in the derivation of new and practically relevant tracking solutions.

## BACKGROUND

The book covers the essential aspects of the derivation and representation of tracking filters using PGFLs. In the past decade, first author Roy Streit brought several aspects of this promising field to the tracking community after the invention of the intensity filter (iFilter) using Poisson point processes (see, for example, [4] and [5]). Therein, Streit lay the foundation of a universal representation of multiobject tracking filters by PGFLs for the design, comparison, and derivation of existing and new, customized tracking filters. Based on the well-understood theory of point processes [6], [7] and the classic publication of Moyal [8], in which finite point processes are first characterized using PGFLs, Streit derived several tracking filter representations and aspects using PGFLs and AC. In particular, [4] and [5] give a first overview of the potential of generating functional representation for the comparison and derivation of multiobject tracking filters. Simple and finite point processes, which are random variables (RVs), represent a set of points. These points are variable in their number and spatial distribution, representing the set of targets, measurements, or clutter. Furthermore, the authors use PGFs and PGFLs to encode the Bayesian probability structure of the point process model. Finally, the well-understood tool of a functional derivative decodes the probabilities and yields the statistics needed to describe and implement the respective tracking filter. This approach is modular and divides the tracking filter into different modeling aspects, which makes it an ideal choice for the design and derivation of new filters. In addition, it is a unique choice to understand and compare existing tracking filters, which makes it an ideal approach for engineers new to the topic of target tracking.

The reviewed book combines aspects of previous publications, adds important details, and provides an excellent didactical structure, starting from the easiest and most accessible example using discrete state spaces, and therefore generating functions, and ending up with a general description of several state-of-the-art filters from past decades. Furthermore, the authors discuss the practical relevance and present options for the efficient numerical implementation of AC-derived tracking filters. The appendix presents the broad mathematical background of the unifying approach without interrupting the didactic concept of the main body. If the reader is interested in further details, the references are exhaustive. An excellent overview and the technical details on AC can be found in [9], the technical details on point processes, which are used to model targets and measurements, can be found in [6] and [7].

## OUTLINE OF ANALYTIC COMBINATORICS FOR MULTIPLE OBJECT TRACKING

The book contains six main chapters plus three additional chapters on the mathematical background in the appendix. In the following, the individual chapters in the main body are briefly discussed.

### CHAPTER 1

In the first chapter, the authors explain how to describe a tracking filter with a single function using the simplest example of a single-target problem: The state of the object of interest is modeled such that it might or might not exist, and a single sensor might or might not produce a single detection of the object on a discrete grid. The state space of the target is discretized, making the possible state estimates countable and enabling the authors to present the idea of the approach by the application of a PGF, the well-known and discrete version of a PGFL, which can be described as "a clothesline on which we hang up a sequence of numbers for display" [10]. Herbert Wilf's quote shows what the idea of generating functions is about: A single function encodes the statistics of the underlying RV in a compact, clear, and comparable way. If needed, the moments, i.e., the statistics of the underlying RV, can be decoded using ordinary differentiation. In this way, the relevant statistical information, which is needed to implement a tracking filter, can be extracted.

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The authors start their description of tracking filters using generating functions after a short introduction to AC, sensor and object models in tracking, likelihood functions, and measurement-to-target assignments in a didactically elegant way. First, the authors model the existence and detection of a target using a Bernoulli RV, that is, they consider the following problem: "Statement A. At most one object exists and, if it does exist, the sensor may or may not generate one measurement" [11]. This simplest example (Statement A) suffices to explain the main idea of the concept of a generating function. In parallel, the reader can study the mathematical foundations of the discussion in Appendix A. This enables the authors to present their concepts without interrupting the common goal to make the main idea clear. Using this simple example, the authors discuss the important topics of a conditional (how does the measurement generating function look if one object exists) and the marginal PGF of the number of measurements. In addition, the authors explain the branching form of a generating function, making the benefit of using AC for the design, derivation, and unification of tracking filters obvious. In the second and third examples, the authors add gridded random measurements and consider the following problem: "Statement B. At most one object exists and, if it does exist, the sensor may or may not generate a random measurement Y" [11]. Afterward, the authors add gridded random object states to the problem, which yields the more advanced tracking-related example: "Statement C. At most one random object exists and, if it does exist in state X, the sensor may or may not generate a random measurement Y" [11]. Afterward, the authors present the PGF for the Bayes theorem and derive it for the three statements, e.g., what does the generating function for the number of existing targets look like if the number of measurements is known? This sets the stage for Bayesian tracking filters. Finally, the authors incorporate different models of object existence and detection (multiple object existence, random number of object existence, and false alarms) into the generating function of the original problem. They give a first impression of how easily an engineer can adapt the generating function and therefore the tracking filter to a specific tracking challenge. Illustrative examples make the considerations clearer and demonstrate the principle of Wilf's clothesline.

## CHAPTER 2

The second chapter sets the stage for the complex multiobject target tracking filters by introducing "filters that track a single target" [11] on a continuous state space, that is, the classical Bayes-Markov [12], probabilistic data association (PDA) [12], and integrated PDA (IPDA) [13] filters. Because most applications presume a continuous target state and/or measurement space, the authors introduce PGFLs. To this end, the authors determine the PGF for the Bayes theorem and take the small cell limit of the two grids that, by becoming infinitesimally small, yields the PGFL for the Bayes theorem on a continuous state and measurement space. Afterward, the authors define the basic PGFL of the Bayes–posterior point process and use it to set up a broad collection of multiobject tracking filters.

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The first example is the Bayes-Markov filter, whose extension yields the PDA filter, which additionally models clutter and optionally gating. Afterward, the authors present the IPDA filter, which integrates a model of object existence into the PGFL and simultaneously demonstrates how the adaption or extension of an existing PGFL is performed intuitively. Finally, the classical Kalman filter [1] is formulated using PGFLs. For each filter, the authors apply the technique of secular functions introduced by Streit in [14] to compute the functional derivatives by ordinary differentiation in order to derive the respective exact Bayesian posterior distribution.

## CHAPTER 3

The third chapter extends the considerations of the second chapter with the introduction of "filters that track a specified number of targets" [11], starting with the well-known extension of the PDA filter, the joint PDA (JPDA) [15]. Because multiple potential measurement origins exist, various measurement-to-target assignment configurations appear. However, in contrast to the traditional derivation of the JPDA filter, which starts by defining all possible measurement-to-target assignments, the AC approach reveals its elegance by first characterizing the statistical properties of measurements and objects. The possible assignments are revealed when the derivatives computed. Thus, the authors demonstrate systematically that AC is "an alternative way to conceptualize these filters combinatorially [sic]" [11]. The authors explain this nontrivial filter derivation and its properties by extending the results of the single-target PDA derivation using AC. An interested reader should be able to follow the arguments of the authors easily (by studying the appendix in parallel) and learn the main ingredients of an AC-tracking filter derivation. The benefit of the AC approach becomes obvious when the authors introduce the famous joint integrated PDA (JIPDA) [16] filter. Modeling track initiation and termination, the number of assignments results in a complex derivation using the traditional path (any engineer who has tried to implement the JIPDA filter will agree on its complexity). However, the AC approach presents the PGFL of the JIPDA in a short and elegant way as the core of the implementation. The complexity of the measurement-to-target assignment becomes apparent when the posterior density is derived using functional derivatives. At a first glance, this seems to be complicated, but the computation of derivatives can be performed automatically using automatic differentiation. A variant of the JPDA filter that models unresolved measurements from two targets demonstrates the potential of the AC approach in the design and adaption of new or existing filters to a specific problem. An illustrative numerical example of the tracking of targets with unresolved measurements using the JPDA filter closes the third chapter.

## CHAPTER 4

The JIPDA has great complexity because each target (and thus its state space) possesses an identifier and because it models the events of track termination and initiation. Therefore, an application of the JIPDA in numerically complex applications usually is restricted. Furthermore, of important relevance for practical applications are filters that "track a variable number of targets" [11]. To this end, the authors apply an efficient concept to the JPDA filter that ignores distinguishable targets; that is, they perform superposition of target state spaces. The lack of information brings a highly applicable multiobject tracking filter to life. Thereby, the concept of AC not only shows its power in the derivation of a new tracking filter by simple operation on the PGFL but also shows relationships between existing filters, like the JPDA with superposition (JPDAS) and the cardinalized probability hypothesis density (CPHD) filter. After introducing state-dependent models for object birth, death, and spawning, the authors show that the famous probability hypothesis density (PHD) filter [17] is a special case of the CPHD filter. Having seen the derivations of tracking filters using superposition, the interested reader should be able to understand further tracking filters (e.g., [18] and [3]) and be ready to derive a problem-specific customized approach. The chapter closes with a numerical efficient particle implementation for superposed AC-derived tracking filters using complex step differentiation [19], [20], [21], which not only makes AC a concept for the design, unification, and understanding of tracking filters but also shows that the concept is interesting for deriving numerical efficient particle–based implementations.

### CHAPTER 5

In an analogous way to Chapter 4 and the relationship between the JPDA and the CPHD filters, the authors demonstrate the relationship between the JIPDA and the multi-Bernoulli (MB) filters using AC. The authors thereby talk about the small but fine difference between the MB filter and the CPHD filter. Each of the Bernoulli models within the MB filter is a sequential detector for a single object (due to the model of target existence within the JIPDA filter), whereas the model of the CPHD filter makes it an estimator of the number of object detection decisions. The authors present variants of the MB filter, including the multi-Bernoulli mixture (MBM) filter originally presented in [22]. The authors discuss several other MB filter variants, including labeled MB and MBM filters, and name a large list of relevant references. The authors apply AC to show that there is a close relationship between "Multi-Bernoulli Mixture and Multi-Hypothesis Tracking filters" [11], [23] for one scan. Finally, the chapter closes with a numerical investigation of the newly derived JIPDA with superposition (JIPDAS) filter.

### CHAPTER 6

In the last chapter, the authors first summarize their results of applying AC to multiobject tracking filters and visualize the connection with the specific filter in a single figure. Afterward, possible extensions of the application of AC to the field of target tracking are discussed, including multiobject trajectory filters, tracking with multiple unresolved objects, and evaluation of further statistics like spatial and temporal correlation. Furthermore, the authors propose two techniques (saddle point approximation [24] and complex step differentiation [19], [20], [21]) for an efficient particle implementation of the filters. Finally, the authors use the alternative way of formulating tracking filters by assignments (see, e.g., [25]) to discuss the applicability of AC to solve integer linear programming problems.

## SUMMARY

The authors give a broad overview of the possibilities and benefits of an application of AC to the world of Bayesian target tracking in their book *Analytic Combinatorics for Multiple Object Tracking* in a didactically excellent way. This book can be recommended to all tracking engineers, independent of whether they are at a beginner or expert level. The list of benefits is long. AC provides the following:

- Unification and representation of tracking filters in a single PGFL
- Understanding of the similarities and differences between existing tracking filters by consideration of a single functional
- Derivation of tracking filters without explicitly enumerating all possible assignments
- The possibility of applying highly efficient numerical implementations
- An adaption and simple, customized design of multiobject tracking filters
- Straightforward evaluation of several summary statistics, like factorial moments
- Reduction of notational burden of multiobject tracking filters

The authors derive the most important tracking filters using AC and use only as much mathematical theory as is essentially necessary to follow the main thread. In addition, all of the mathematical theory the reader needs for the details is contained in the appendix, and the authors name useful references to understand the mathematical theory. After reading the book, readers can design their own tracking filters using AC, compare them in a unified fashion with existing filters, and efficiently implement particle versions of the filters. Overall, the book is a pleasure to read and shows its relevance in several publications on the topic of AC for target tracking.

***"After reading the book, readers can design their own tracking filters using AC, compare them in a unified fashion with existing filters, and efficiently implement particle versions of the filters."***

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