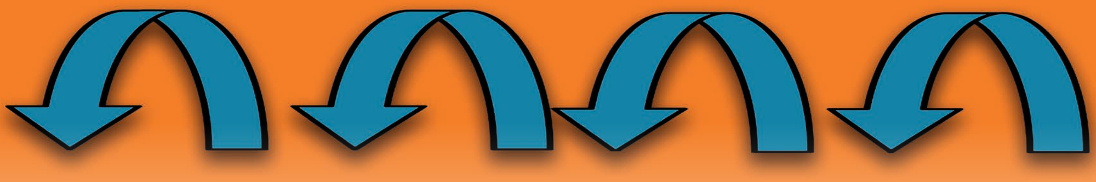

State and Trajectory Estimation Using Accumulated State Densities

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Abstract—In tracking and sensor data fusion applications, the full information on kinematic object properties accumulated over a certain discrete time window up to the present time is contained in the conditional joint probability density function of the kinematic state vectors referring to each time step in this window. This density is conditioned by the time series of all sensor data collected at the present time and has accordingly been called an accumulated state density (ASD). ASDs provide a unified treatment of filtering and retrodiction insofar as by marginalizing them appropriately, the standard filtering and retrodiction densities are obtained. In addition, ASDs fully describe the posterior correlations between the states at different instants of time. Therefore, the closed-form solution of ASDs are directly connected to many real-world problems. This article presents an overview of the applications such as out-of-sequence processing, smoothing, distributed filtering, and batch processing.

ON THE PROBLEM OF TRACKING OBJECTS

To a degree never known before, decision makers in a net-centric world have access to vast amounts of data. For effective use of this information potential in real-world applications, however, the data streams must not overwhelm the decision makers involved. On the contrary, the data must be fused in such a way that high-quality information for situation pictures results, the basis for decision making.

Situation pictures are produced by spatiotemporally processing various pieces of sensor information that in themselves often have only limited value for understanding the underlying situation. In this context, object "tracks" are of particular importance [1], [2], [3]. Tracking faces an omnipresent aspect in real-world application insofar as it is dealing with fusion of data produced at different instants of time; i.e., tracking is important in all applications in which a particular emphasis is placed on sensor data given by time series.

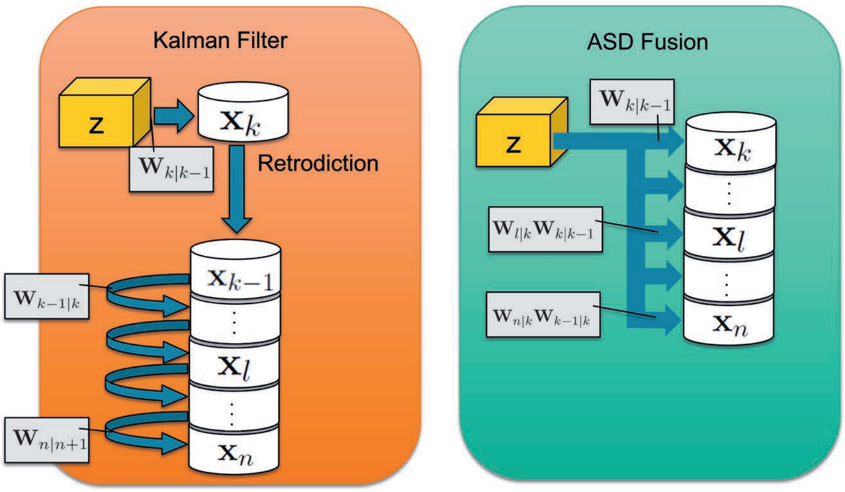

Figure 1
Schematic comparison of ASD and Kalman filter fusion.

In most tracking algorithms, the characteristics of conditional probability densities $p(\mathbf{x}_l \,|\, Z^k)$ of (joint) object states $\mathbf{x}_l$ are calculated, which describe the available knowledge of the object properties at a certain instant of time t_l, given a time series Z^k of imperfect sensor data accumulated up to time t_k. In certain applications, however, the kinematic object states $\mathbf{x}_k,\ldots,\mathbf{x}_n$, $n \leq k$,[1] accumulated over a certain time window from a past instant of time t_n up to the present time t_k is of interest. The statistical properties of the accumulated state vectors are completely described by the joint probability density function (pdf) of them, $p(\mathbf{x}_k,\ldots,\mathbf{x}_n \,|\, Z^k)$, which is conditioned by the time series Z^k. These densities may be called accumulated state densities (ASDs) [4]. By marginalizing them, the standard filtering and retrodiction densities directly result; in other words, ASDs provide a unified description of filtering and retrodiction as shown schematically

[1] Note that in the notation used here, the newest state $\mathbf{x}_k$ comes first, then older states in time-reversed order.

in Figure 1. In addition, ASDs fully describe the correlations between the state estimates at different instants of time.

In [5], for example, ASDs are considered to provide a more comprehensive treatment of issues in particle filtering. To some extent, the notion of ASDs might be considered as a step backward insofar as in the old days of object tracking it was known that one could express a linear-Gaussian estimation problem in a joint, i.e., "accumulated" fashion, while Kalman's approach was a way to find a recursive solution. Nevertheless, as shown in this article, it is useful for various tracking applications to have a *recursive* algorithm to find the parameters of an ASD.

In this article, the closed-form solution for the ASD posterior density is provided together with a couple of algorithms for various applications. The applications discussed in this article are processing of out-of-sequence (OoS) measurements, smoothing, batch processing of measurements, and distributed estimation.

This article is organized as follows. The Bayesian Tracking Paradigm section summarizes basic facts of the Bayesian tracking paradigm. In the Notion of ASD section, the ASD is introduced along with a discussion of closed formulae for the parameters of the ASD in the case in which Kalman filtering can be applied to tracking. The use of ASDs for solving the various applications, such as OoS processing, smoothing, batch processing, and distributed tracking, are the topic of Selected Applications of ASDs section. Algorithms are provided within each section.

THE BAYESIAN TRACKING PARADIGM

A Bayesian tracking algorithm is an iterative updating scheme for calculating conditional pdfs $p(\mathbf{x}_l \mid Z^k)$ that represent all available knowledge on the object states $\mathbf{x}_l$ at discrete instants of time t_l. The densities are explicitly conditioned by the sensor data Z^k accumulated up to some time t_k, typically the present time. Implicitly, they are also determined by all available context knowledge on the sensor characteristics, the dynamical object properties, the environment of the objects, topographical maps, or tactical rules governing the objects' overall behavior.

With respect on the instant of time t_l at which estimates of the object states $\mathbf{x}_l$ are required, the related density iteration process is referred to as *prediction* ($t_l > t_k$), *filtering* ($t_l = t_k$), or *retrodiction* ($t_l < t_k$). The propagation of the probability densities involved is given by three basic update equations.

PREDICTION

The prediction density $p(\mathbf{x}_k \mid Z^{k-1})$ is obtained by combining the evolution model $p(\mathbf{x}_k \mid \mathbf{x}_{k-1})$ with the previous filtering density $p(\mathbf{x}_{k-1} \mid Z^{k-1})$:

$$p\left(\mathbf{x}_{k-1} \mid Z^{k-1}\right) \xrightarrow[\text{constraints}]{\text{evolution model}} p\left(\mathbf{x}_k \mid Z^{k-1}\right) \tag{1}$$

$$p\left(\mathbf{x}_k \mid Z^{k-1}\right) = \int \mathrm{d}\mathbf{x}_{k-1}\, \underbrace{p\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}\right)}_{\text{evolution model}} \underbrace{p\left(\mathbf{x}_{k-1} \mid Z^{k-1}\right)}_{\text{previous filtering}}.$$

FILTERING

The filtering density $p(\mathbf{x}_k \mid Z^k)$ is obtained by combining the sensor model $p(Z_k \mid \mathbf{x}_k)$, also called the "likelihood function," with the prediction density $p(\mathbf{x}_k \mid Z^{k-1})$ according to

$$p\left(\mathbf{x}_k \mid Z^{k-1}\right) \xrightarrow[\text{sensor model}]{\text{current sensor data}} p\left(\mathbf{x}_k \mid Z^k\right) \tag{2}$$

$$p\left(\mathbf{x}_k \mid Z^k\right) = \frac{p\left(Z_k \mid \mathbf{x}_k\right) p\left(\mathbf{x}_k \mid Z^{k-1}\right)}{\int \mathrm{d}\mathbf{x}_k\, \underbrace{p\left(Z_k \mid \mathbf{x}_k\right)}_{\text{sensor model}} \underbrace{p\left(\mathbf{x}_k \mid Z^{k-1}\right)}_{\text{prediction}}}.$$

RETRODICTION

The retrodiction density $p(\mathbf{x}_l \mid Z^k)$ is obtained by combining the object evolution model $p(\mathbf{x}_{l+1} \mid \mathbf{x}_l)$ with the previous prediction and filtering densities according to:

$$p\left(\mathbf{x}_l \mid Z^k\right) \xleftarrow[\text{evolution model}]{\text{filtering, prediction}} p\left(\mathbf{x}_{l+1} \mid Z^k\right) \tag{3}$$

$$p\left(\mathbf{x}_l \mid Z^k\right) = \int \mathrm{d}\mathbf{x}_{l+1} \frac{\overbrace{p\left(\mathbf{x}_{l+1} \mid \mathbf{x}_l\right)}^{\text{evolution}} \overbrace{p\left(\mathbf{x}_l \mid Z^l\right)}^{\text{prev. filtering}}}{\underbrace{p\left(\mathbf{x}_{l+1} \mid Z^l\right)}_{\text{prev. prediction}}} \underbrace{p\left(\mathbf{x}_{l+1} \mid Z^k\right)}_{\text{prev. retrodiction}}.$$

Being the natural antonym of prediction, the technical term retrodiction was introduced by Oliver Drummond in a series of papers [6]. Adopting the standard terminology [7], we could speak of *fixed-interval* retrodiction.

According to this paradigm, an *object track* represents all relevant knowledge on a time varying object state of interest, including its history and measures that describe the quality of this knowledge. As a technical term, "track" is therefore either a synonym for the collection of densities $p(\mathbf{x}_l \mid Z^k)$, $l = 1, \ldots, k, \ldots$, or of suitably chosen parameters characterizing them, such as estimates related to appropriate risk functions and the corresponding estimation error covariance matrices.

NOTION OF ASD

All information on the object states accumulated over a time window $t_k, t_{k-1}, \ldots, t_n$ of length $k - n + 1$,

$$\mathbf{x}_{k:n} = \left(\mathbf{x}_k^\top, \ldots, \mathbf{x}_n^\top\right)^\top \tag{4}$$

that can be extracted from the time series of accumulated sensor data Z^k up to and including time t_k is contained in a joint density function $p(\mathbf{x}_{k:n} \mid Z^k)$, which may be called ASD. Here, t_k typically denotes the current time, and $t_n \le t_k$ is the time of initialization or the lower bound of a sliding time window. Via marginalizing over $\mathbf{x}_k, \ldots, \mathbf{x}_{l+1}, \mathbf{x}_{l-1}, \ldots, \mathbf{x}_n$,

$$p\left(\mathbf{x}_l \mid Z^k\right) = \int \mathrm{d}\mathbf{x}_k, \ldots, \mathrm{d}\mathbf{x}_{l+1}, \mathrm{d}\mathbf{x}_{l-1}, \ldots, \mathrm{d}\mathbf{x}_n \; p\left(\mathbf{x}_k, \ldots, \mathbf{x}_n \mid Z^k\right), \tag{5}$$

the filtering density $p(\mathbf{x}_k \mid Z^k)$ for $l = k$ and the retrodiction densities $p(\mathbf{x}_l \mid Z^k)$ for $l < k$ result from the ASD. ASDs, thus, in a way unify the notions of filtering and retrodiction. In addition, ASDs also contain all mutual correlations between the individual object states at different instants of time. Bayes' theorem provides a recursion formula for updating ASDs:

$$p\left(\mathbf{x}_{k:n} \mid Z^k\right) = \frac{p\left(Z_k \mid \mathbf{x}_k\right) p\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}\right) p\left(\mathbf{x}_{k-1:n} \mid Z^{k-1}\right)}{\int \mathrm{d}\mathbf{x}_{k:n}\; p\left(Z_k \mid \mathbf{x}_k\right) p\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}\right) p\left(\mathbf{x}_{k-1:n} \mid Z^{k-1}\right)}. \tag{6}$$

The sensor data Z_k from time t_k explicitly appears in this representation. A little formalistically speaking, "sensor data processing" means nothing else than to achieve by certain reformulations that the sensor data is no longer explicitly present.

Under conditions in which Kalman filtering is applicable (perfect data sensor-data-to-track association, linear Gaussian sensor, and evolution models), a closed-form representation of $p(\mathbf{x}_{k:n} \mid Z^k)$ can be derived. In this case, let the likelihood function be given by

$$p\left(Z_k \mid \mathbf{x}_k\right) = \mathcal{N}\left(z_k; \mathbf{H}_k \mathbf{x}_k, \mathbf{R}_k\right), \tag{7}$$

where $Z_k = \mathbf{z}_k$ denotes the vector of sensor measurements at time t_k, $\mathbf{x}_k = \mathbf{x}_k$ the kinematic state vector of the object, $\mathbf{H}_k$ the measurement matrix, and $\mathbf{R}_k$ the measurement error covariance matrix, while the Markovian evolution model of the object is represented by

$$p\left(\mathbf{x}_k \mid \mathbf{x}_{k-1}\right) = \mathcal{N}\left(\mathbf{x}_k; \mathbf{F}_{k|k-1} \mathbf{x}_{k-1}, \mathbf{Q}_{k|k-1}\right) \tag{8}$$

with an evolution matrix $\mathbf{F}_{k|k-1}$ and a corresponding evolution covariance matrix $\mathbf{Q}_{k|k-1}$. For given initial knowledge $p\left(\mathbf{x}_n \mid Z^n\right) = \mathcal{N}\left(\mathbf{x}_n; \mathbf{x}_{n|n}, \mathbf{P}_{n|n}\right)$ and for $k = 1, 2, \ldots$, the filtered parameters are the result of the well-known prediction-filtering recursion given by

$$p\left(\mathbf{x}_{k+1} \mid Z^k\right) = \mathcal{N}\left(\mathbf{x}_{k+1}; \mathbf{x}_{k+1|k}, \mathbf{P}_{k+1|k}\right) \tag{9}$$

$$\mathbf{x}_{k+1|k} = \mathbf{F}_{k+1|k} \mathbf{x}_{k|k} \tag{10}$$

$$\mathbf{P}_{k+1|k} = \mathbf{F}_{k+1|k} \mathbf{P}_{k|k} \mathbf{F}_{k+1|k}^{\top} + \mathbf{Q}_{k+1|k}, \tag{11}$$

and

$$p\left(\mathbf{x}_k \mid Z^k\right) = \mathcal{N}\left(\mathbf{x}_k; \mathbf{x}_{k|k}, \mathbf{P}_{k|k}\right) \tag{12}$$

$$\mathbf{x}_{k|k} = \begin{cases} \mathbf{x}_{k|k-1} + \mathbf{W}_{k|k-1}\left(\mathbf{z}_k - \mathbf{H}_k \mathbf{x}_{k|k-1}\right) \\ \mathbf{P}_{k|k}\left(\mathbf{P}_{k|k-1}^{-1} \mathbf{x}_{k|k-1} + \mathbf{H}_k^{\top} \mathbf{R}_k^{-1} \mathbf{z}_k\right) \end{cases} \tag{13}$$

$$\mathbf{P}_{k|k} = \begin{cases} \mathbf{P}_{k|k-1} - \mathbf{W}_{k|k-1} \mathbf{S}_{k|k-1} \mathbf{W}_{k|k-1}^{\top} \\ \left(\mathbf{P}_{k|k-1}^{-1} + \mathbf{H}_k^{\top} \mathbf{R}_k^{-1} \mathbf{H}_k\right)^{-1}. \end{cases} \tag{14}$$

There exist two equivalent formulations of the Kalman update formulae according to the two versions of the product formula (74). The innovation covariance matrix $\mathbf{S}_{l|l-1}$ and the *Kalman gain* matrix at some given time t_l are given by

$$\mathbf{S}_{l|l-1} = \mathbf{H}_l \mathbf{P}_{l|l-1} \mathbf{H}_l^{\top} + \mathbf{R}_l. \tag{15}$$

$$\mathbf{W}_{l|l-1} = \mathbf{P}_{l|l-1} \mathbf{H}_l^{\top} \mathbf{S}_{l|l-1}^{-1}. \tag{16}$$

Because ASD states $\mathbf{x}_k, \ldots \mathbf{x}_n$ are conditioned on the full data Z^k up to time t_k, its mean and covariance is directly related to the result of the well-known Rauch–Tung–Striebel (RTS) recursion

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|l} + \mathbf{W}_{l|l+1}\left(\mathbf{x}_{l+1|k} - \mathbf{x}_{l+1|l}\right) \tag{17}$$

$$\mathbf{P}_{l|k} = \mathbf{P}_{l|l} + \mathbf{W}_{l|l+1}\left(\mathbf{P}_{l+1|k} - \mathbf{P}_{l+1|l}\right) \mathbf{W}_{l|l+1}^{\top}, \tag{18}$$

and a "retrodiction gain" matrix

$$\mathbf{W}_{l|l+1} = \mathbf{P}_{l|l} \mathbf{F}_{l+1|l}^{\top} \mathbf{P}_{l+1|l}^{-1}. \tag{19}$$

Now, using the abbreviation

$$\mathbf{D}_{l|k} = \mathbf{P}_{l|k} - \mathbf{W}_{l|l+1} \mathbf{P}_{l+1|k} \mathbf{W}_{l|l+1}^{\top}. \tag{20}$$

the closed-form solution of the ASD posterior in the linear-Gaussian case is given by the multivariate normal distribution

$$p\left(\mathbf{x}_{k:n} \mid Z^k\right) = \mathcal{N}\left(\mathbf{x}_{k:n}; \mathbf{x}_{k:n|k}, \mathbf{P}_{k:n|k}\right), \tag{21}$$

with a joint expectation vector $\mathbf{x}_{k:n|k}$ defined by

$$\mathbf{x}_{k:n|k} = \left(\mathbf{x}_{k|k}^{\top}, \mathbf{x}_{k-1|k}^{\top}, \ldots, \mathbf{x}_{n|k}^{\top}\right)^{\top}, \tag{22}$$

while the corresponding joint covariance matrix $\mathbf{P}_{k:n|k}$ can be written as an inverse of a tridiagonal block matrix that is given in (23). The tridiagonal structure is a consequence of the Markov property of the underlying evolution model. This representation of the inverse of $\mathbf{P}_{k:n|k}$ is useful in practical calculations.

By a repeated use of the matrix inversion lemma (see the Appendix) and an induction argument, the inverse of this tridiagonal block matrix can be calculated. The resulting block matrix is given in (24). For this representation, the following abbreviations were used:

$$\mathbf{P}^{-1}_{k:n|k} = \begin{pmatrix} \mathbf{T}_{k|k} & -\mathbf{W}^{\top}_{k-1|k}\mathbf{D}^{-1}_{k-1|k} & \mathbf{O} & \cdots & \mathbf{O} \\ -\mathbf{D}^{-1}_{k-1|k}\mathbf{W}_{k-1|k} & \mathbf{T}_{k-1|k} & -\mathbf{W}^{\top}_{k-2|k}\mathbf{Q}^{-1}_{k-2|k} & \ddots & \vdots \\ \mathbf{O} & -\mathbf{D}^{-1}_{k-2|k}\mathbf{W}_{k-2|k} & \ddots & \ddots & \mathbf{O} \\ \vdots & \ddots & \ddots & \mathbf{T}_{n+1|k} & -\mathbf{W}^{\top}_{n|k}\mathbf{D}_{n|k} \\ \mathbf{O} & \cdots & \mathbf{O} & -\mathbf{D}_{n|k}\mathbf{W}_{n|k} & \mathbf{T}_{n|k} \end{pmatrix}. \tag{23}$$

$$\mathbf{P}_{k:n|k} = \begin{pmatrix} \mathbf{P}_{k|k} & \mathbf{P}_{k|k}\mathbf{W}^{\top}_{k-1|k} & \mathbf{P}_{k|k}\mathbf{W}^{\top}_{k-2|k} & \cdots & \mathbf{P}_{k|k}\mathbf{W}^{\top}_{n|k} \\ \mathbf{W}_{k-1|k}\mathbf{P}_{k|k} & \mathbf{P}_{k-1|k} & \mathbf{P}_{k-1|k}\mathbf{W}^{\top}_{k-2|k-1} & * & \mathbf{P}_{k-1|k}\mathbf{W}^{\top}_{n|k-1} \\ \mathbf{W}_{k-2|k}\mathbf{P}_{k|k} & \mathbf{W}_{k-2|k-1}\mathbf{P}_{k-1|k} & \mathbf{P}_{k-2|k} & * & \vdots \\ \vdots & * & * & * & \mathbf{P}_{n+1|k}\mathbf{W}^{\top}_{n|n+1} \\ \mathbf{W}_{n|k}\mathbf{P}_{k|k} & \mathbf{W}_{n|k-1}\mathbf{P}_{k-1|k} & \cdots & \mathbf{W}_{n|n+1}\mathbf{P}_{n+1|k} & \mathbf{P}_{n|k} \end{pmatrix}, \tag{24}$$

$$\mathbf{W}_{l|k} = \prod_{\lambda=l}^{k-1} \mathbf{W}_{\lambda|\lambda+1} = \prod_{\lambda=l}^{k-1} \mathbf{P}_{\lambda|\lambda}\mathbf{F}^{\top}_{\lambda+1|\lambda}\mathbf{P}^{-1}_{\lambda+1|\lambda}. \tag{25}$$

The densities $\left\{\mathcal{N}\left(\mathbf{x}_l;\mathbf{x}_{l|k},\mathbf{P}_{l|k}\right)\right\}_{l=n}^{k}$ are directly obtained via marginalizing, because the covariance matrices $\mathbf{P}_{l|k}$, $n \leq l \leq k$, appear on the diagonal of this block matrix. Note that the ASD is completely defined by the results of prediction, filtering, and retrodiction obtained for the time window $t_k, \ldots, t_n$, i.e., it is a by-product for Kalman filtering and RTS smoothing. It is not surprising that the smoothed estimates and the error covariances appear as the block entries in the mean and the block covariance, respectively. However, the interesting result is to have a closed-form solution for the structure of the joint covariance matrix. In particular, the cross covariances of states at different instants of time can be taken from the off-diagonal entries of $\mathbf{P}_{k:n|k}$.

RECURSIVE ASD FILTER IMPLEMENTATION

The following sections show how to iteratively calculate the parameters $\mathbf{x}_{k:n|k}$ and $\mathbf{P}_{k:n|k}$. A short summary for a straightforward implementation is provided in Table 1. Assume the posterior ASD at time t_{k-1} is given in terms of $\mathbf{x}_{k-1:n|k-1}$ and $\mathbf{P}_{k-1:n|k-1}$. The prediction of the state is straightforward due to the Markov proposition:

$$\mathbf{x}_{k:n|k-1} = \left(\mathbf{x}^{\top}_{k|k-1} \quad \mathbf{x}^{\top}_{k-1|k-1} \quad \cdots \quad \mathbf{x}^{\top}_{n|k-1}\right)^{\top}, \tag{26}$$

where $\mathbf{x}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{x}_{k-1|k-1}$ is equivalent to a Kalman filter prediction. For the ASD covariance prediction, a recursive formulation of the ASD covariance in (24) is used:

$$\mathbf{P}_{k:n|k-1} = \begin{pmatrix} \mathbf{P}_{k|k-1} & \mathbf{P}_{k|k-1}\mathbf{W}^{\top}_{k-1:n} \\ \mathbf{W}_{k-1:n}\mathbf{P}_{k|k-1} & \mathbf{P}_{k-1:n|k-1} \end{pmatrix}, \tag{27}$$

where

$$\mathbf{P}_{k|k-1} = \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}^{\top}_{k|k-1} + \mathbf{Q}_{k|k-1}, \tag{28}$$

$$\mathbf{W}_{k-1:n} = \begin{pmatrix} \mathbf{W}_{k-1|k} \\ \mathbf{W}_{k-2:n}\mathbf{W}_{k-1|k} \end{pmatrix}. \tag{29}$$

Table 1

Recursive ASD Algorithm

Initialization	Set $\mathbf{x}_{0\|0}, \mathbf{P}_{0\|0}$.
Prediction $t_0 \to t_1$	$\mathbf{x}_{1\|0} = \mathbf{F}_{1\|0}\mathbf{x}_{0\|0}$ $\mathbf{P}_{1\|0} = \mathbf{F}_{1\|0}\mathbf{P}_{0\|0}\mathbf{F}^{\top}_{1\|0} + \mathbf{Q}_{1\|0}$ $\mathbf{x}_{1:0\|0} = \left(\mathbf{x}^{\top}_{1\|0}\mathbf{x}^{\top}_{0\|0}\right)^{\top}$ $\mathbf{P}_{1:0\|0} = \begin{pmatrix} \mathbf{P}_{1\|0} & \mathbf{F}_{1\|0}\mathbf{P}_{0\|0} \\ \mathbf{P}_{0\|0}\mathbf{F}^{\top}_{1\|0} & \mathbf{P}_{0\|0} \end{pmatrix}$
Filtering $\mathbf{z}_1$ at t_1	$\Pi_1 = (\mathbf{I}, \mathbf{O})$ $\mathbf{S}_1 = \mathbf{H}_1\Pi_1\mathbf{P}_{1:0\|0}\Pi^{\top}_1\mathbf{H}^{\top}_1 + \mathbf{R}_1$ $\mathbf{W}_{1:0\|0} = \mathbf{P}_{1:0\|0}\Pi^{\top}_1\mathbf{H}^{\top}_1\mathbf{S}^{-1}_1$ $\mathbf{x}_{1:0\|1} = \mathbf{x}_{1:0\|0} + \mathbf{W}_{1:0\|0}\left(\mathbf{z}_1 - \mathbf{H}_1\Pi_1\mathbf{x}_{1:0\|0}\right)$ $\mathbf{P}_{1:0\|1} = \mathbf{P}_{1:0\|0} - \mathbf{W}_{1:0\|0}\mathbf{S}_1\mathbf{W}^{\top}_{1:0\|0}$
Prediction $t_{k-1} \to t_k$	$\mathbf{x}_{k\|k-1} = \mathbf{F}_{k\|k-1}\mathbf{x}_{k-1\|k-1}$ $\mathbf{P}_{k\|k-1} = \mathbf{F}_{k\|k-1}\mathbf{P}_{k-1\|k-1}\mathbf{F}^{\top}_{k\|k-1} + \mathbf{Q}_{k\|k-1}$ $\mathbf{x}_{k:0\|k-1} = \left(\mathbf{x}^{\top}_{k\|k-1}\mathbf{x}^{\top}_{k-1:0\|k-1}\right)^{\top}$ $\mathbf{P}^{(k-1)}_{k-1:n\|k-1} = \begin{pmatrix} \mathbf{P}_{k-1\|k-1} \\ \mathbf{W}_{k-2\|k-1}\mathbf{P}_{k-1\|k-1} \\ \vdots \\ \mathbf{W}_{0\|k-1}\mathbf{P}_{k-1\|k-1} \end{pmatrix}$ $\mathbf{P}_{k:n\|k-1} = \begin{pmatrix} \mathbf{P}_{k\|k-1} & \mathbf{F}_{k\|k-1}\left(\mathbf{P}^{(k-1)}_{k-1:n\|k-1}\right)^{\top} \\ \mathbf{P}^{(k-1)}_{k-1:n\|k-1}\mathbf{F}^{\top}_{k\|k-1} & \mathbf{P}_{k-1:n\|k-1} \end{pmatrix}$
Filtering $\mathbf{z}_k$ at t_k	$\Pi_k = (\mathbf{I}, \mathbf{O}, \ldots, \mathbf{O})$ $\mathbf{S}_k = \mathbf{H}_k\Pi_k\mathbf{P}_{k:0\|k-1}\Pi^{\top}_k\mathbf{H}^{\top}_k + \mathbf{R}_k$ $\mathbf{W}_{k:0\|k-1} = \mathbf{P}_{k:0\|k-1}\Pi^{\top}_k\mathbf{H}^{\top}_k\mathbf{S}^{-1}_k$ $\mathbf{x}_{k:0\|k} = \mathbf{x}_{k:0\|k-1} + \mathbf{W}_{k:0\|k-1}\left(\mathbf{z}_k - \mathbf{H}_k\Pi_k\mathbf{x}_{k:0\|k-1}\right)$ $\mathbf{P}_{k:0\|k} = \mathbf{P}_{k:0\|k-1} - \mathbf{W}_{k:0\|k-1}\mathbf{S}_k\mathbf{W}^{\top}_{k:0\|k-1}$
Sliding Window	Prune estimate for t_n from $\mathbf{x}_{k:n\|n}$.
	Prune column and row for t_n from $\mathbf{P}_{k:n\|n}$.

The expression in (27) can be simplified to

$$\mathbf{P}_{k:n|k-1} = \begin{pmatrix} \mathbf{P}_{k|k-1} & \mathbf{F}_{k|k-1}\left(\mathbf{P}^{(k-1)}_{k-1:n|k-1}\right)^{\top} \\ \mathbf{P}^{(k-1)}_{k-1:n|k-1}\mathbf{F}^{\top}_{k|k-1} & \mathbf{P}_{k-1:n|k-1} \end{pmatrix}, \tag{30}$$

where $\mathbf{P}^{(k-1)}_{k-1:n|k-1}$ represents the $(k-1)$th block column for $n = 1$.

For the filtering step, it is assumed that the prior parameters $\mathbf{x}_{k:n|k-1}$ and $\mathbf{P}_{k:n|k-1}$ are given. As the measurement error is assumed to be independent from the past, the sensor likelihood function can be expressed by an application of projections Π_k onto the current state:

$$p\left(\mathbf{z}_k \mid \mathbf{x}_k\right) = p\left(\mathbf{z}_k \mid \Pi_k\mathbf{x}_{k:n}\right) \tag{31}$$

$$= \mathcal{N}\left(\mathbf{z}_k; \mathbf{H}_k \Pi_k \mathbf{x}_{k:n}, \mathbf{R}_k\right), \tag{32}$$

where

$$\Pi_k = \left(\mathbf{I}, \mathbf{O}, \ldots, \mathbf{O}\right). \tag{33}$$

In the previous notation, $\mathbf{I}$ is an identity matrix in the dimension of the state, and $\mathbf{O}$ is the corresponding zero matrix. Then, the posterior parameters are obtained by the multiplication of the local prior density and the likelihood function. An application of the product formula in the Appendix yields

$$\mathbf{x}_{k:n|k} = \mathbf{x}_{k:n|k-1} + \mathbf{W}_{k:n|k-1}\left(\mathbf{z}_k - \mathbf{H}_k \Pi_k \mathbf{x}_{k:n|k-1}\right), \tag{34}$$

$$\mathbf{P}_{k:n|k} = \mathbf{P}_{k:n|k-1} - \mathbf{W}_{k:n|k-1} \mathbf{S}_k \mathbf{W}_{k:n|k-1}^\top \tag{35}$$

$$\mathbf{W}_{k:n|k-1} = \mathbf{P}_{k:n|k-1} \Pi_k^\top \mathbf{H}_k^\top \mathbf{S}_k^{-1}, \tag{36}$$

$$\mathbf{S}_k = \mathbf{H}_k \Pi_k \mathbf{P}_{k:n|k-1} \Pi_k^\top \mathbf{H}_k^\top + \mathbf{R}_k. \tag{37}$$

Note that the dimension of $\mathbf{S}_k$ is in the dimension of $\mathbf{z}_k$, which is small in most applications. Moreover, as stated previously, the smoothed states and covariances, respectively, are obtained by a single update step.

SELECTED APPLICATIONS OF ASDS

SMOOTHING

If the estimation of the trajectory of all states at multiple instants of time is of interest, smoothing or retrodiction has to be applied. The updated states conditioned on the complete set of sensor data up to time t_k can be obtained by the RTS equations (17) and (18). This, however, requires a second recursion that has to be initiated after each filtering step. By using the block line version of the ASD update in (34) and (35), it can easily be seen that the smoothed state that refers to time t_l is given by the equations

$$\mathbf{x}_{l|k} = \mathbf{x}_{l|k-1} + \mathbf{W}_{l|k} \mathbf{W}_{k|k-1}\left(\mathbf{z}_k - \mathbf{H}_k \mathbf{x}_{k|k-1}\right), \tag{38}$$

$$\mathbf{P}_{l|k} = \mathbf{P}_{l|k-1} - \mathbf{W}_{l|k} \mathbf{W}_{k|k-1} \mathbf{S}_{k|k-1} \mathbf{W}_{k|k-1}^\top \mathbf{W}_{l|k}^\top. \tag{39}$$

Because $\mathbf{W}_{k|k-1}(\mathbf{z}_k - \mathbf{H}_k \mathbf{x}_{k|k-1})$ is part of the Kalman filter update and that

$$\mathbf{W}_{l|k} = \mathbf{W}_{l|l+1} \mathbf{W}_{l+1|k}, \tag{40}$$

this smoothing algorithm was shown to be significantly faster than the standard RTS retrodiction [8], [9], [10].

BATCH PROCESSING

If initial conditions are given by the pdf $p\left(\mathbf{x}_n\right) = \mathcal{N}\left(\mathbf{x}_n; \mathbf{x}_{n|0}, \mathbf{P}_{n|0}\right)$ and a set of measurements $\mathbf{z}_{k:n} = \{\mathbf{z}_k, \mathbf{z}_{k-1}, \ldots, \mathbf{z}_n\}$ is to be processed, the ASD formulae directly provide the means to estimate the trajectory $\mathbf{x}_{k:n}$ on the basis of the sensor data. This can easily be seen by an application of the Bayes' theorem:

$$p\left(\mathbf{x}_{k:n} \mid Z^k\right) = \frac{p\left(\mathbf{z}_{k:n} \mid \mathbf{x}_{k:n}\right) p\left(\mathbf{x}_{k:n}\right)}{\int \mathrm{d}\mathbf{x}_{k:n}\, p\left(\mathbf{z}_{k:n} \mid \mathbf{x}_{k:n}\right) p\left(\mathbf{x}_{k:n}\right)}, \tag{41}$$

where $p(\mathbf{x}_{k:n})$ is conditioned on the initial knowledge on $\mathbf{x}_n$. Therefore, it can be obtained by a successive application of the ASD prediction (26) and (30) on the initial pdf. This yields a Gaussian density

$$p\left(\mathbf{x}_{k:n}\right) = \mathcal{N}\left(\mathbf{x}_{k:n}; \mathbf{x}_{k:n|0}, \mathbf{P}_{k:n|0}\right) \tag{42}$$

where the mean is given by

$$\mathbf{x}_{k:n|0} = \begin{pmatrix} \mathbf{x}_{k|0}^\top & \mathbf{x}_{k-1|0}^\top & \cdots & \mathbf{x}_{n|0}^\top \end{pmatrix}^\top \tag{43}$$

and $\mathbf{P}_{k:n|0}$ is given in (47) and the following abbreviations

$$\mathbf{x}_{l|0} = \mathbf{F}_{l|0} \mathbf{x}_{n|0} = \mathbf{F}_{l|l-1} \mathbf{x}_{l-1|0} \tag{44}$$

$$\mathbf{P}_{l|0} = \mathbf{F}_{l|0} \mathbf{P}_{n|0} \mathbf{F}_{l|0}^\top + \mathbf{Q}_{l|0} \tag{45}$$

$$= \mathbf{F}_{l|l-1} \mathbf{P}_{l-1|0} \mathbf{F}_{l|l-1}^\top + \mathbf{Q}_{l|l-1} \tag{46}$$

$$\mathbf{P}_{k:n|0} = \begin{pmatrix} \mathbf{P}_{k|0} & \mathbf{F}_{k|k-1}\mathbf{P}_{k-1|0} & \cdots & & \mathbf{F}_{k|k-1}\cdots\mathbf{F}_{n+1|n}\mathbf{P}_{n|0} \\ \mathbf{P}_{k-1|0}\mathbf{F}_{k|k-1}^\top & \ddots & & & \\ \vdots & & \mathbf{P}_{n+2|0} & \mathbf{F}_{n+2|n+1}\mathbf{P}_{n+1|0} & \mathbf{F}_{n+2|n+1}\mathbf{F}_{n+1|n}\mathbf{P}_{n|0} \\ \vdots & & \mathbf{P}_{n+1|0}\mathbf{F}_{n+2|n+1}^\top & \mathbf{P}_{n+1|0} & \mathbf{F}_{n+1|n}\mathbf{P}_{n|0} \\ \mathbf{P}_{n|0}\mathbf{F}_{n+1|n}^\top\cdots\mathbf{F}_{k|k-1}^\top & \cdots & \mathbf{P}_{n|0}\mathbf{F}_{n+1|n}^\top\mathbf{F}_{n+2|n+1}^\top & \mathbf{P}_{n|0}\mathbf{F}_{n+1|n}^\top & \mathbf{P}_{n|0} \end{pmatrix} \tag{47}$$

were used. Because the measurements are mutually conditionally independent, the likelihood of the accumulated measurement set $\mathbf{z}_{k:n}$ is given by

$$p\left(\mathbf{z}_{k:n} \mid \mathbf{x}_{k:n}\right) = \mathcal{N}\left(\mathbf{z}_{k:n}; \mathbf{H}_{k:n} \mathbf{x}_{k:n}, \mathbf{R}_{k:n}\right) \tag{48}$$

where

$$\mathbf{z}_{k:n} = \begin{pmatrix} \mathbf{z}_k^\top & \cdots & \mathbf{z}_n^\top \end{pmatrix}^\top, \tag{49}$$

$$\mathbf{R}_{k:n} = \operatorname{blkdiag}\begin{pmatrix} \mathbf{R}_k & \cdots & \mathbf{R}_n \end{pmatrix}, \tag{50}$$

$$\mathbf{H}_{k:n} = \operatorname{blkdiag}\begin{pmatrix} \mathbf{H}_k & \cdots & \mathbf{H}_n \end{pmatrix}. \tag{51}$$

According to Bayes' theorem an application of the product formula directly yields the fully *filtered and smoothed* trajectory is given by the posterior ASD

$$p\left(\mathbf{x}_{k:n} \mid Z^k\right) = \mathcal{N}\left(\mathbf{x}_{k:n}; \mathbf{x}_{k:n|k}, \mathbf{P}_{k:n|k}\right), \tag{52}$$

where

$$\mathbf{x}_{k:n|k} = \mathbf{x}_{k:n|0} + \mathbf{W}_{k:n|0}\left(\mathbf{z}_{k:n} - \mathbf{H}_{k:n}\mathbf{x}_{k:n|0}\right), \quad (53)$$

$$\mathbf{P}_{k:n|k} = \mathbf{P}_{k:n|0} - \mathbf{W}_{k:n|0}\mathbf{S}_{k:n}\mathbf{W}_{k:n|0}^{\top}, \quad (54)$$

$$\mathbf{W}_{k:n|0} = \mathbf{P}_{k:n|0}\mathbf{H}_{k:n}^{\top}\mathbf{S}_{k:n}^{-1}, \quad (55)$$

$$\mathbf{S}_{k:n} = \mathbf{H}_{k:n}\mathbf{P}_{k:n|0}\mathbf{H}_{k:n}^{\top} + \mathbf{R}_{k:n}. \quad (56)$$

This algorithm is summarized in Table 2. Moreover, for each instant of time t_l, there exists a weighting matrix $\mathbf{K}_{l\leftarrow n}$ and a set of matrices $\{\mathbf{L}_{l\leftarrow j}\}_{j=n}^{k}$ such that the smoothed estimate for time t_l is given by

$$\mathbf{x}_{l|k} = \mathbf{K}_{l\leftarrow n}\mathbf{x}_{n|0} + \sum_{j=n}^{k}\mathbf{L}_{l\leftarrow j}\mathbf{z}_{j}. \quad (57)$$

This can be considered as the block line version of the ASD update in (53). The proof and all details can be found in [11].

OOS PROCESSING

In many real-world applications of sensor data fusion, one has to be aware of OoS measurements. Due to latencies in the underlying communication infrastructure, for example, such measurements arrive at a processing node in a distributed data fusion system "too late," i.e., after sensor data with a later time stamp have already been processed.

Consider a measurement $\mathbf{z}_m$ produced at time t_m with $n \le m$, i.e., possibly before the "present" time t_k, where the time series Z^k is available and has been exploited. It is now required to compute the impact of this new but late sensor information has on the present and the past target states $\mathbf{x}_l$, $l \in \{n, \ldots, k\}$, i.e. on the accumulated target state $\mathbf{x}_{k:n}$. Let $\mathbf{z}_m$ be a measurement of the target state $\mathbf{x}_m$ at time t_m characterized by a Gaussian likelihood function, which is defined by a measurement matrix $\mathbf{H}_m$ and a corresponding measurement error covariance matrix $\mathbf{R}_m$. Furthermore, it is useful to renumber the target states $\mathbf{x}_k, \ldots, \mathbf{x}_n$ such that $\mathbf{x}_k, \ldots, \mathbf{x}_m, \ldots, \mathbf{x}_n =: \mathbf{x}_{k:m:n}$ are consistent with their time stamps $(t_l)_{l=k,\ldots,m,\ldots,n}$.

To process the measurement $\mathbf{z}_m$, the prior ASD

$$p\left(\mathbf{x}_{k:m:n} \mid Z^k \setminus \{\mathbf{z}_m\}\right) = \mathcal{N}\left(\mathbf{x}_{k:m:n}; \mathbf{x}_{k:m:n|k}, \mathbf{P}_{k:m:n|k}\right) \quad (58)$$

is required. The parameters of this density are given by the closed-form formulae for ASDs by using the *continuous-time retrodiction* [10] for the mean and covariance of the state at time t_m:

$$\mathbf{x}_{m|k} = \mathbf{x}_{m|m-1} + \mathbf{W}_{m+1|m}\left(\mathbf{x}_{m+1|k} - \mathbf{x}_{m+1|m-1}\right), \quad (59)$$

$$\mathbf{P}_{m|k} = \mathbf{P}_{m|m-1} + \mathbf{W}_{m+1|m}\cdot\left(\mathbf{P}_{m+1|k} - \mathbf{P}_{m+1|m-1}\right)\mathbf{W}_{m+1|m}^{\top}, \quad (60)$$

$$\mathbf{x}_{m|m-1} = \mathbf{F}_{m|m-1}\mathbf{x}_{m-1|k}, \quad (61)$$

Table 2

Batch Processing Algorithm	
Initialization	Set $\mathbf{x}_{n\|0}, \mathbf{P}_{n\|0}$, gather $\{\mathbf{z}_l\}_{l=n}^{k}$
Prior up to Time t_k	For $l = n+1, \ldots, k$ compute
	$\mathbf{x}_{l\|0} = \mathbf{F}_{l\|l-1}\mathbf{x}_{l-1\|0}$
	$\mathbf{P}_{l\|0} = \mathbf{F}_{l\|l-1}\mathbf{P}_{l-1\|0}\mathbf{F}_{l\|l-1}^{\top} + \mathbf{Q}_{l\|l-1}$
	End for
	$\mathbf{x}_{k:n\|0} = \left(\mathbf{x}_{k\|0}^{\top} \;\; \mathbf{x}_{k-1\|0}^{\top} \;\; \ldots \;\; \mathbf{x}_{n\|0}^{\top}\right)^{\top}$
	$\mathbf{P}_{k:n\|0}$ as in (47).
Batch Likelihood	$\mathbf{z}_{k:n} = \left(\mathbf{z}_k^{\top} \;\; \ldots \;\; \mathbf{z}_n^{\top}\right)^{\top}$,
	$\mathbf{R}_{k:n} = \text{blkdiag}(\mathbf{R}_k \;\; \ldots \;\; \mathbf{R}_n)$,
	$\mathbf{H}_{k:n} = \text{blkdiag}(\mathbf{H}_k \;\; \ldots \;\; \mathbf{H}_n)$.
Processing	$\mathbf{x}_{k:n\|k} = \mathbf{x}_{k:n\|0} + \mathbf{W}_{k:n\|0}\left(\mathbf{z}_{k:n} - \mathbf{H}_{k:n}\mathbf{x}_{k:n\|0}\right)$,
	$\mathbf{P}_{k:n\|k} = \mathbf{P}_{k:n\|0} - \mathbf{W}_{k:n\|0}\mathbf{S}_{k:n}\mathbf{W}_{k:n\|0}^{\top}$,
	$\mathbf{W}_{k:n\|0} = \mathbf{P}_{k:n\|0}\mathbf{H}_{k:n}^{\top}\mathbf{S}_{k:n}^{-1}$,
	$\mathbf{S}_{k:n} = \mathbf{H}_{k:n}\mathbf{P}_{k:n\|0}\mathbf{H}_{k:n}^{\top} + \mathbf{R}_{k:n}$.

$$\mathbf{P}_{m|m-1} = \mathbf{F}_{m|m-1}\mathbf{P}_{m-1|k}\mathbf{F}_{m|m-1}^{\top} + \mathbf{Q}_{m|m-1}. \quad (62)$$

In terms of the ASD state, the measurement $\mathbf{z}_m$ now is "in sequence" and can be processed by means of the ASD update equations (34) and (35). To this end, the projection onto state $\mathbf{x}_k$ has to be replaced by a projection onto $\mathbf{x}_m$.

The block line approach for the exact OoS processing is described in [11]. In this article, it is shown that the estimate for time t_l, $l \in \{n, \ldots, k\}$ is given by

$$\mathbf{W}_{l,m|k} = \mathbf{W}_{l|m}\mathbf{P}_{\max\{l,m\}|k}\mathbf{W}_{m|l}^{\top}\mathbf{H}_m^{\top}\mathbf{S}_{m|k}^{-1}. \quad (63)$$

$$\mathbf{x}_{l|k,m} = \mathbf{x}_{l|k} + \mathbf{W}_{l,m|k}\left(\mathbf{z}_m - \mathbf{H}_m\mathbf{x}_{m|k}\right), \quad (64)$$

$$\mathbf{P}_{l|k,m} = \mathbf{P}_{l|k} - \mathbf{W}_{l,m|k}\mathbf{S}_{m|k}\mathbf{W}_{l,m|k}^{\top}. \quad (65)$$

Here, the notation is extended such that $\mathbf{W}_{l|j}$ is the identity matrix for $j \le l$.

DISTRIBUTED ASD FUSION

Often in multisensor applications, it is required to preprocess data at each sensor node to economize on bandwidth. The preprocessed parameters are then fused to the global estimate. In the literature, there exist plenty of algorithms, many of which are derived from different approaches. To provide an entire overview of those is beyond the scope of this article; therefore, we refer the reader to [2] and [3] for examples. An exact method for the linear-Gaussian case is the distributed Kalman filter (DKF) [12] and [13], i.e., the resulting fused track is equivalent to a centralized Kalman filter processing all measurements from every sensor.

The DKF, however, requires sensor models to be known to each processing node. In [14] and [15], it is shown that ASDs can be used for exact distributed fusion without the sensor parameters being globally known. This is of particular importance for applications with nonlinear measurement functions, because the linearized measurement error becomes data dependent. This is achieved by exploiting the fact that for S sensors it holds that

$$\mathcal{N}\left(\mathbf{x}_k;\mathbf{F}_{k|k-1}\mathbf{x}_{k-1},\mathbf{Q}_{k|k-1}\right) \propto \mathcal{N}\left(\mathbf{x}_k;\mathbf{F}_{k|k-1}\mathbf{x}_{k-1},S\mathbf{Q}_{k|k-1}\right)^S. \tag{66}$$

By means of a "spread" process noise covariance matrix $S\mathbf{Q}_{k|k-1}$, one achieves a product representation of *local* ASDs:

$$p\left(\mathbf{x}_{k:n} \mid Z^k\right) \propto \prod_{s=1}^{S} \mathcal{N}\left(\mathbf{x}_{k:n};\mathbf{x}^s_{k:n|k},\mathbf{P}^s_{k:n|k}\right). \tag{67}$$

Here, the parameters $\mathbf{x}^s_{k:n|k}$ and $\mathbf{P}^s_{k:n|k}$ are obtained from the closed-form solution for ASDs by using the relaxed evolution model on the right side of (66) and local data from sensor s only [14]. As a consequence, the fusion of these local parameters becomes an almost trivial convex combination:

$$\mathbf{x}_{k:n|k} = \mathbf{P}_{k:n|k} \sum_{s=1}^{S} \left(\mathbf{P}^s_{k:n|k}\right)^{-1} \mathbf{x}^s_{k:n|k} \tag{68}$$

$$\mathbf{P}_{k:n|k} = \left(\sum_{s=1}^{S} \left(\mathbf{P}^s_{k:n|k}\right)^{-1}\right)^{-1}. \tag{69}$$

This fusion rule is exact whenever the full ASD parameters are transmitted to the fusion center. By truncating the time series of estimates, an approximation is achieved that yields close to optimal results [13]. For the implementation, the interested reader can easily follow the summary in Table 3.

ASDs can also be used to perform *multistep tracklet fusion* [13]. Because the *equivalent measurements*

$$\mathbf{Y}^s_{k:n} = \left(\mathbf{P}^s_{k:n|k}\right)^{-1} - \left(\mathbf{P}^s_{k:n|n}\right)^{-1} \tag{70}$$

$$\mathbf{y}_{k:n} = \left(\mathbf{P}^s_{k:n|k}\right)^{-1} \mathbf{x}^s_{k:n|k} - \left(\mathbf{P}^s_{k:n|n}\right)^{-1} \mathbf{x}^s_{k:n|n} \tag{71}$$

are mutually independent when conditioned on the ASD state, they can be used to update a centralized ASD track:

$$\mathbf{P}_{k:n|k} = \left(\left(\mathbf{P}_{k:n|n}\right)^{-1} + \sum_{s=1}^{S} \mathbf{Y}^s_{k:n}\right)^{-1}, \tag{72}$$

$$\mathbf{x}_{k:n|k} = \mathbf{P}_{k:n|k}\left(\left(\mathbf{P}_{k:n|n}\right)^{-1}\mathbf{x}_{k:n|n} + \sum_{s=1}^{S} \mathbf{y}^s_{k:n}\right). \tag{73}$$

Table 3

Distributed ASD Filter

Initialization	For $s = 1, \ldots, S$ compute $\left\{\left(\mathbf{x}^s_{k:n|k}, \mathbf{P}^s_{k:n|k}\right)\right\}$
Prediction	As in Table 1, except that $\mathbf{Q}_{l|l-1}$ is replaced by $S\mathbf{Q}_{l|l-1}$ for all l.
Filtering	As in Table 1, with $\mathbf{z}_k = \mathbf{z}^s_k$ for each processing node s.
Fusion Rule	$\mathbf{x}_{k:n|k} = \mathbf{P}_{k:n|k} \sum_{s=1}^{S} \left(\mathbf{P}^s_{k:n|k}\right)^{-1} \mathbf{x}^s_{k:n|k}$
	$\mathbf{P}_{k:n|k} = \left(\sum_{s=1}^{S} \left(\mathbf{P}^s_{k:n|k}\right)^{-1}\right)^{-1}$.

Note that the ASDs in (70) and (71) are *not* based on the relaxed evolution model, i.e., they are optimal with respect to the local sensor data.

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APPENDIX: PRODUCT FORMULA

For matrices of suitable dimensions, the following formula for products of Gaussians holds:

$$\mathcal{N}(\mathbf{z};\mathbf{Hx},\mathbf{R})\mathcal{N}(\mathbf{x};\mathbf{y},\mathbf{P}) = \mathcal{N}(\mathbf{z};\mathbf{Hy},\mathbf{S})\begin{cases}\mathcal{N}(\mathbf{x};\mathbf{y}+\mathbf{W}\nu,\mathbf{P}-\mathbf{WSW}^\top)\\ \mathcal{N}(\mathbf{x};\mathbf{Q}(\mathbf{P}^{-1}\mathbf{y}+\mathbf{H}^\top\mathbf{R}^{-1}\mathbf{z}),\mathbf{Q})\end{cases} \tag{74}$$

with the following abbreviations:

$$\nu = \mathbf{z} - \mathbf{Hy} \tag{75}$$

$$\mathbf{S} = \mathbf{HPH}^\top + \mathbf{R} \tag{76}$$

$$\mathbf{W} = \mathbf{PH}^\top\mathbf{S}^{-1} \tag{77}$$

$$\mathbf{Q}^{-1} = \mathbf{P}^{-1} + \mathbf{H}^\top\mathbf{R}^{-1}\mathbf{H}. \tag{78}$$

Wolfgang Koch studied physics and mathematics at the Aachen Technical Universit, Germany, where he earned a Dr. rer. nat. degree in theoretical physics. For many years, he was head of the Department of Sensor Data and Information Fusion at Fraunhofer FKIE, an institute of the Fraunhofer Society, the largest institution for applied research in Europe. On various topics within the area of sensor data and information fusion, he has published a well-referenced textbook on target tracking and sensor data fusion, 16 handbook chapters, and well above 200 journal and conference articles. He is one of the coeditors of the two volume handbook *Novel Radar Techniques and Applications*.

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