Misassociation Probability in M2TA and T2TA

JAVIER ARETA
YAAKOV BAR-SHALOM
RON ROTHROCK

This paper presents procedures to calculate the probability that the measurement or the track originating from an extraneous target will be (mis)associated with a target of interest for the cases of Nearest Neighbor and Global association. For the measurement-to-track (M2T) case, it is shown that these misassociation probabilities depend, under certain assumptions, on a particular—covariance weighted—norm of the difference between the targets' predicted measurements. For the Nearest Neighbor M2T association, the exact solution, obtained for the case of equal track covariances, is based on a noncentral chi-square distribution. An approximate solution is also presented for the case of unequal track prediction covariances. For the Global M2T association case an approximation is presented for the case of "similar" track covariances. In the general case of unequal track covariances where this approximation fails, a more complicated but exact method based on the inversion of the characteristic function is presented. The track-to-track (T2T) association case involves correlated random variables for which the exact probability density function is very hard to obtain. Moment matching approximations are used that provide very accurate results. The theoretical results, confirmed by Monte Carlo simulations, quantify the benefit of Global vs. Nearest Neighbor M2T association. These results are applied to problems of single sensor as well as centralized fusion architecture multiple sensor tracking.

Manuscript received October 3, 2007; revised May 7, 2008; released for publication June 16, 2008.

Refereeing of this contribution was handled by Yvo Boers.

Authors' addresses: J. Areta and Y. Bar-Shalom, Univ. of Connecticut, ECE Dept., Storrs, CT 06269-2157, E-mail: (areta@ee.uconn.edu), (ybs@ee.uconn.edu); R. Rothrock, Sparta, Arlington, VA 22203, E-mail: (Ron_Rothrock@rosslyn.sparta.com).

1557-6418/07/$17.00

1. INTRODUCTION

This paper deals with the closed form misassociation probability formula for measurement-to-track association (M2T) and track-to-track association (T2T). The emphasis of this work is in closely spaced targets, which is much more prevalent in the real world than association of clutter to target tracks. Thus clutter is not considered in the sequel. In the first sections we develop the procedure to calculate the probability that the measurement associated by a likelihood based assignment algorithm to a target of interest originates from another (extraneous) target as a function of the state estimates and covariances of the tracks. Both a Nearest Neighbor[1] (NN) as well as Global[2] (G) assignment are considered. An approximate procedure is developed for the T2T association, as a closed form of the probability density function is very hard to find, due to the existing correlation between the track estimates. These closed form expressions should be useful when the knowledge of the performance of a system is to be quantified, for example, in the selection of a radar given it accuracy and the expected scenarios it could encounter. Also as in [6], it could be used to predict the number of measurements needed to achieve a certain performance. The model used for the targets is deterministic—they are located at a certain separation distance in the measurement space, expressed in terms of the track state estimates mapped into the measurement space. The association problem[3] was investigated in [10] for a different model, namely, the targets were assumed randomly distributed (i.i.d. uniform in a hyperball of a sufficiently large radius). Extensive work on the association of tracks from two sources, using kinematic, feature and classification information was done in [14, 7]. In [9] a more complex T2TA problem accounting for registration errors and mismatch in the number of tracks is considered. To obtain meaningful results, the track model considered is simplistic, assuming isotropic errors of the same variance. The model considered here allows performance evaluation of association algorithms under more realistic conditions, namely, with arbitrary measurement prediction covariances and the results are expressed in terms of the target separation distance.

Section 2 formulates the M2T association problem. The calculation of the misassociation probability for a Nearest Neighbor association is described in Section 3

[1] Strictly speaking, this is local NN because it considers the association of only one measurement to a track at a time (see [1] Section 3.2), and the measurement/track with smaller association cost is assigned to it. The other measurement is associated to the remaining track.

[2] This considers simultaneously all the measurements and tracks so the assignment is chosen as the association pair with overall smaller cost, and, unlike the local NN, has a unique solution [11].

[3] Two-dimensional (2-D, also known as "single frame"), i.e., between two lists—the list of measurements from the latest scan/frame and the list of tracks. Association is called sometimes "correlation"; since correlation has a well defined meaning in probability/statistics, we will not use it for association.

for the situation where the innovation covariances of the two tracks are equal as well as a generalization for unequal innovation covariances case. Section 4 introduces the Global association criterion, and the probabilities of misassociation for the two innovation covariance cases are obtained. The case of T2T association is considered in Section 5. Simulation results presented in Section 6 compare the theoretical calculations with Monte Carlo runs. Conclusions are presented in Section 7.

2. FORMULATION OF THE M2T ASSOCIATION PROBLEM

The predicted measurements (at the current time, not indicated for simplicity) for the two targets are denoted as $\hat{z}_i$, with associated covariances S_i, $i = 1,2$. These covariances are detailed in the sequel.

The pdf of the measurement prediction from the target of interest, designated as 1, is

$$p(z_1) = \mathcal{N}(z_1; \hat{z}_1, S_1) \tag{1}$$

while the pdf of the measurement prediction from the extraneous target, designated as 2, is

$$p(z_2) = \mathcal{N}(z_2; \hat{z}_2, S_2). \tag{2}$$

It is assumed that the assignment algorithm, using the likelihood function (or likelihood ratio) as a criterion, will associate to target t the measurement whose likelihood of having originated from target t is the largest. The likelihood of measurement z_i having originated from target t is given by the pdf of a measurement originating from target (track) t—the predicted measurement pdf—evaluated at z_i, namely,

$$\begin{aligned}\Lambda_{it} &= P_{D_t}\mathcal{N}(z_i; \hat{z}_t, S_{it}) \\ &\triangleq P_{D_t}|2\pi S_{it}|^{-1/2}\exp[-\tfrac{1}{2}(z_i - \hat{z}_t)' S_{it}^{-1}(z_i - \hat{z}_t)]\end{aligned} \tag{3}$$

where P_{D_t} is the detection probability of target t, and

$$S_{it} = H_t P_t H_t' + R_i \tag{4}$$

where H_t is the measurement matrix for track t and R_i is the measurement noise covariance for z_i. Since this noise covariance can be a function of the SNR, it has the index of the measurement. The likelihood ratio of originating from this track vs. from (random) clutter is this likelihood function divided by a constant, which is the spatial density of the clutter, assumed Poisson distributed [4].

Thus the index of the measurement that will be associated with track t is [1]

$$\begin{aligned}i^*(t) &= \arg\max_i [P_{D_t}\mathcal{N}(z_i; \hat{z}_t, S_{it})] \\ &= \arg\min_i [(z_i - \hat{z}_t)' S_{it}^{-1}(z_i - \hat{z}_t) + \ln|2\pi S_{it}|].\end{aligned} \tag{5}$$

Note that the target detection probability does not appear in the final expression above because all the likelihoods of association with track t have the same multiplier. In the case that the innovation covariance matrices are equal[4]

$$S_{it} = S_t \qquad \forall \quad i \tag{6}$$

then the log of the determinant of the covariance matrix in (5) is the same for all i and

$$i^*(t) = \arg\min_i [(z_i - \hat{z}_t)' S_t^{-1}(z_i - \hat{z}_t)]. \tag{7}$$

Consequently, under assumption (6), for track 1 the associated measurement report (AMR) will be the one whose normalized (Mahalanobis) distance squared to $\hat{z}_1$, given by

$$D(z, \hat{z}_1) = (z - \hat{z}_1)' S_1^{-1}(z - \hat{z}_1) \tag{8}$$

is the smallest. This amounts to a "local Nearest Neighbor" (designated as NN) assignment. Therefore, the misassociation event ($\mathrm{MA}_{21}^{\mathrm{NN}}$) that the measurement from target 2 is assigned to track 1 (which represents target 1) occurs if

$$\{\mathrm{MA}_{21}^{\mathrm{NN}}\} \triangleq \{D(z_2, \hat{z}_1) < D(z_1, \hat{z}_1)\}. \tag{9}$$

The analysis of misassociation in the case of a global assignment (G) will be presented later.

3. NEAREST NEIGHBOR M2T MISASSOCIATION

3.1. Equal Innovation Covariances

This section evaluates the probability of misassociation of the NN assignment technique which considers tracks independently under a simplifying assumption. Assuming that

$$z_1 \sim \mathcal{N}(\hat{z}_1, S_1) \tag{10}$$

the pdf of $D(z_1, \hat{z}_1)$ is chi-square with n_z (dimension of z) degrees of freedom (d.o.f.), to be denoted as

$$p_{D(z_1,\hat{z}_1)}(x) = \chi^2_{n_z}(x). \tag{11}$$

To obtain the pdf of the "competition," $D(z_2, \hat{z}_1)$, it is rewritten as

$$\begin{aligned}D(z_2, \hat{z}_1) &= (z_2 - \hat{z}_1)' S_1^{-1}(z_2 - \hat{z}_1) \\ &= (z_2 - \hat{z}_2 + \hat{z}_2 - \hat{z}_1)' S_1^{-1}(z_2 - \hat{z}_2 + \hat{z}_2 - \hat{z}_1).\end{aligned} \tag{12}$$

Note that the above contains, in addition to the deterministic quantity $\hat{z}_2 - \hat{z}_1$, the difference $z_2 - \hat{z}_2$. The latter is random with covariance S_2, but the quadratic form in (12) contains the matrix S_1.

As shown below, the pdf of (12) is noncentral chi-square if the matrix in the quadratic form is the covariance of $z_2 - \hat{z}_2$. Consequently, it will be first assumed

[4]This holds approximately when a single sensor tracks two close targets.

that[5]

$$S_1 = S_2 = S. \tag{13}$$

Using the Cholesky decomposition of S^{-1}

$$S^{-1} = (S^{-1/2})' S^{-1/2} \tag{14}$$

one can rewrite (12) as

$$\begin{aligned} D(z_2,\hat{z}_1) = {} & [S^{-1/2}(z_2 - \hat{z}_2) + S^{-1/2}(\hat{z}_2 - \hat{z}_1)]' \\ & \times [S^{-1/2}(z_2 - \hat{z}_2) + S^{-1/2}(\hat{z}_2 - \hat{z}_1)]. \end{aligned} \tag{15}$$

Denoting the n_z-vector

$$\xi_{21} \stackrel{\Delta}{=} [S^{-1/2}(z_2 - \hat{z}_2) + S^{-1/2}(\hat{z}_2 - \hat{z}_1)] \tag{16}$$

the distance (15) is its norm squared, i.e.,

$$D(z_2,\hat{z}_1) = \xi'_{21}\xi_{21} = \sum_{i=1}^{n_z} (\xi_{21}(i))^2. \tag{17}$$

Since

$$\mathrm{cov}[\xi_{21}] = S^{-1/2} S (S^{-1/2})' = I \tag{18}$$

the components $\xi_{21}(i)$ of ξ_{21} are independent Gaussian random variables with nonzero means and unity variance. Thus

$$\xi_{21}(i) \sim \mathcal{N}(\bar{\xi}_{21}(i), 1), \qquad i = 1,\ldots,n_z. \tag{19}$$

where $\bar{\xi}_{21}(i)$ it the i-th element of $[S^{-1/2}(\hat{z}_2 - \hat{z}_1)]$, $i = 1,\ldots,n_z$.

Consequently [15], the pdf of (17) is noncentral chi-square with n_z d.o.f. and non-centrality parameter

$$\begin{aligned} \lambda &= \sum_{1}^{n_z} ([S^{-1/2}(\hat{z}_2 - \hat{z}_1)]_i)^2 \\ &= [S^{-1/2}(\hat{z}_2 - \hat{z}_1)]'[S^{-1/2}(\hat{z}_2 - \hat{z}_1)] \\ &= (\hat{z}_2 - \hat{z}_1)' S^{-1} (\hat{z}_2 - \hat{z}_1). \end{aligned} \tag{20}$$

This pdf is denoted as

$$p_{D(z_2,\hat{z}_1)}(x) = \chi^2_{n_z,\lambda}(x). \tag{21}$$

The cumulative distribution function (cdf) corresponding to the above will be denoted as $X^2_{n_z,\lambda}(x)$ and a routine for its evaluation (to be needed below) is available from [5].

The probability of the misassociation event (9) is then given by

$$\begin{aligned} \mathsf{P}_{\mathrm{MA}_{21}^{\mathrm{NN}}} &= \mathsf{P}\{D(z_2,\hat{z}_1) < D(z_1,\hat{z}_1)\} \\ &= \int_0^\infty \mathsf{P}\{D(z_2,\hat{z}_1) < x\} p_{D(z_1,\hat{z}_1)}(x)dx \\ &= \int_0^\infty X^2_{n_z,\lambda}(x)\chi^2_{n_z}(x)dx. \end{aligned} \tag{22}$$

[5]For simplicity, the single indexing of covariances as in (1)–(2) is used in the sequel.

3.2. Unequal Innovation Covariances

This section evaluates the probability of misassociation of the NN assignment technique which considers tracks independently in the general case. While assumption (13) is somewhat limiting, it is not unreasonable to assume that two targets in the same neighborhood have the same state estimation covariance. If (13) is not satisfied, then (16) has to be replaced by

$$\zeta_{21} \stackrel{\Delta}{=} [S_1^{-1/2}(z_2 - \hat{z}_2) + S_1^{-1/2}(\hat{z}_2 - \hat{z}_1)] \tag{23}$$

and

$$D(z_2,\hat{z}_1) = \zeta'_{21}\zeta_{21} = \sum_{i=1}^{n_z} (\zeta_{21}(i))^2. \tag{24}$$

The covariance of (23) is

$$\mathrm{cov}[\zeta_{21}] = S_1^{-1/2} S_2 S_1^{-1/2} \neq I \tag{25}$$

i.e., its components are not independent anymore and (24) is not chi-square distributed. Consequently, one cannot use anymore (22) to evaluate the probability of the misassociation event (9).

In this case the exact distribution of (24) is needed. However, this is not known because the covariance of $z_2 - \hat{z}_2$ is S_2 but the norm is w.r.t. $S_1 \neq S_2$. A moment matching technique will be used to approximate its distribution.

Considering only the zero-mean part of (23), its norm squared is

$$D_0(z_2,\hat{z}_1) = [S_1^{-1/2}(z_2 - \hat{z}_2)]'[S_1^{-1/2}(z_2 - \hat{z}_2)] \tag{26}$$

and its mean is

$$\begin{aligned} E[D_0(z_2,\hat{z}_1)] &= E[[S_1^{-1/2}(z_2 - \hat{z}_2)]'[S_1^{-1/2}(z_2 - \hat{z}_2)]] \\ &= \mathrm{tr}[S_1^{-1} S_2]. \end{aligned} \tag{27}$$

Under the equal covariance assumption, the above would have been equal to n_z (the trace of the $n_z \times n_z$ identity matrix, as in (18)).

Based on the above observation, we will scale $D(z_2,\hat{z}_1)$ to match its mean to what it would have been in the equal covariance case (i.e., n_z) as follows

$$D^*(z_2,\hat{z}_1) \stackrel{\Delta}{=} \alpha_{21} D(z_2,\hat{z}_1) \tag{28}$$

where

$$\alpha_{21} \stackrel{\Delta}{=} \frac{n_z}{\mathrm{tr}[S_1^{-1} S_2]} \tag{29}$$

and will approximate the distribution of $D^*(z_2,\hat{z}_1)$ as noncentral chi-square with n_z d.o.f. and noncentrality parameter

$$\begin{aligned} \lambda^* &= \alpha_{21} \sum_{i=1}^{n_z} ([S_1^{-1/2}(\hat{z}_2 - \hat{z}_1)]_i)^2 \\ &= \alpha_{21} (\hat{z}_2 - \hat{z}_1)' S_1^{-1} (\hat{z}_2 - \hat{z}_1). \end{aligned} \tag{30}$$

The probability of the misassociation event (9) is then given by

$$
\begin{aligned}
\mathsf{P}_{\mathrm{MA}_{21}^{\mathrm{NN}}} &= \mathsf{P}\{D(z_2,\hat{z}_1) < D(z_1,\hat{z}_1)\} \\
&= \mathsf{P}\left\{\frac{D^*(z_2,\hat{z}_1)}{\alpha_{21}} < D(z_1,\hat{z}_1)\right\} \\
&= \int_0^\infty \mathsf{P}\{D^*(z_2,\hat{z}_1) < \alpha_{21}x\} p_{D(z_1,\hat{z}_1)}(x)dx \\
&= \int_0^\infty X^2_{n_z,\lambda^*}(\alpha_{21}x)\chi^2_{n_z}(x)dx. \qquad (31)
\end{aligned}
$$

Since the above is an approximation, its quality will be evaluated via Monte Carlo runs in Section 6.

4. M2T MISASSOCIATION IN A GLOBAL ASSIGNMENT

If a Global assignment [2] is used, then a misassociation (swap of measurements for two targets) occurs if

$$
D_{21} + D_{12} < D_{11} + D_{22} \qquad (32)
$$

since, as in (5), the covariance determinants cancel. This assumes the same noise covariances for the two measurements and the same detection probabilities for the two targets. The evaluation of the probability of this event is done next. Note that the distance term $D(z_i,\hat{z}_j)$ is now noted as D_{ij}, for simplicity. No "gating" [1] (i.e., infinite gating threshold) is assumed because it would require truncated pdfs, which would complicate the analysis and would make little difference in the results because the gates are, typically, above 99%.

The inequality (32) is rewritten so that the random variables in it (z_1, z_2) are each on one side only, namely,

$$
D_{21} - D_{22} < D_{11} - D_{12}. \qquad (33)
$$

The l.h.s. of the above is

$$
\begin{aligned}
D_{21} - D_{22} &= (z_2 - \hat{z}_1)' S_1^{-1}(z_2 - \hat{z}_1) - (z_2 - \hat{z}_2)' S_2^{-1}(z_2 - \hat{z}_2) \\
&= z_2'(S_1^{-1} - S_2^{-1})z_2 - z_2' S_1^{-1}\hat{z}_1 - \hat{z}_1' S_1^{-1} z_2 \\
&\quad + z_2' S_2^{-1}\hat{z}_2 + \hat{z}_2' S_2^{-1} z_2 + \hat{z}_1' S_1^{-1}\hat{z}_1 - \hat{z}_2' S_2^{-1}\hat{z}_2. \qquad (34)
\end{aligned}
$$

As before, this equation should be analyzed for the case of equal covariances, where cancelation of the first term occurs, and for different covariances.

4.1. Global Assignment Misassociation with Equal Innovation Covariances

When both covariances are equal,[6] the quadratic term in (34) vanishes, thus the distribution of $D_{21} - D_{22}$ is Gaussian since

$$
\begin{aligned}
D_{21} - D_{22} &= -2(\hat{z}_1 - \hat{z}_2)' S^{-1} z_2 + \hat{z}_1' S^{-1}\hat{z}_1 - \hat{z}_2' S^{-1}\hat{z}_2 \\
&\triangleq c' z_2 + b \triangleq \Delta_{21}. \qquad (35)
\end{aligned}
$$

Similarly, the term $D_{11} - D_{12}$ can be written as

$$
\begin{aligned}
D_{11} - D_{12} &= -2(\hat{z}_1 - \hat{z}_2)' S^{-1} z_1 + \hat{z}_1' S^{-1}\hat{z}_1 - \hat{z}_2' S^{-1}\hat{z}_2 \\
&\triangleq c' z_1 + b \triangleq \Delta_{12}. \qquad (36)
\end{aligned}
$$

Note that these two Gaussian random variables (RV) are independent because each depends on only one of the measurements. Thus, as they are Gaussian and independent, their difference $\Delta_{21} - \Delta_{12}$ is also Gaussian

$$
\Delta_{21} - \Delta_{12} \sim \mathcal{N}(2(\hat{z}_1 - \hat{z}_2)' S^{-1}(\hat{z}_1 - \hat{z}_2), 8(\hat{z}_1 - \hat{z}_2)' S^{-1}(\hat{z}_1 - \hat{z}_2)) \qquad (37)
$$

so the probability of the misassociation event $\mathrm{MA}^{\mathrm{G}}_{21,12} = \{\Delta_{21} < \Delta_{12}\}$ can be calculated in terms of the cumulative density function Φ of a standard Normal random variable as

$$
\begin{aligned}
\mathsf{P}_{\mathrm{MA}^{\mathrm{G}}_{21,12}} &= \mathsf{P}\{\Delta_{21} - \Delta_{12} < 0\} \\
&= \mathsf{P}\left\{\frac{\Delta_{21} - \Delta_{12} - 2(\hat{z}_1 - \hat{z}_2)' S^{-1}(\hat{z}_1 - \hat{z}_2)}{[8(\hat{z}_1 - \hat{z}_2)' S^{-1}(\hat{z}_1 - \hat{z}_2)]^{1/2}} \right. \\
&\qquad \left. < \frac{-2(\hat{z}_1 - \hat{z}_2)' S^{-1}(\hat{z}_1 - \hat{z}_2)}{[8(\hat{z}_1 - \hat{z}_2)' S^{-1}(\hat{z}_1 - \hat{z}_2)]^{1/2}}\right\} \\
&= \Phi\left(-\left[\frac{(\hat{z}_1 - \hat{z}_2)' S^{-1}(\hat{z}_1 - \hat{z}_2)}{2}\right]^{1/2}\right). \qquad (38)
\end{aligned}
$$

4.2. Global Assignment Misassociation with Unequal Innovation Covariances

In the case that the covariances are different, the quadratic term in (34) does not vanish. Its contribution is higher when the covariance matrices are very different. Thus two approaches to approximate the distribution of the difference will be investigated. The first one approximates the distribution as a noncentral chi-square, using the first two moments, and is expected to provide better results when the quadratic term dominates and has positive eigenvalues. The other approach is to fit a Gaussian with matched mean and variance, and is expected to work better when the contribution of the quadratic term is small.

4.2.1. Moment matching approaches

For the first approach, denote

$$
A_{21} \triangleq S_1^{-1} - S_2^{-1} \qquad (39)
$$

$$
b_{21} \triangleq S_1^{-1}\hat{z}_1 - S_2^{-1}\hat{z}_2 \qquad (40)
$$

$$
c_{21} \triangleq \hat{z}_1' S_1^{-1}\hat{z}_1 - \hat{z}_2' S_2^{-1}\hat{z}_2 \qquad (41)
$$

[6] A similar approach has been taken in [10] assuming, however, random location of the targets according to a spatial Poisson process. In our case the probability of error is a function of a normalized distance between the targets.

one has, by completing the quadratic form

$$
\begin{aligned}
D_{21} - D_{22} &= z_2' A_{21} z_2 - z_2' b_{21} - b_{21}' z_2 + c_{21} \\
&= (z_2 - A_{21}^{-1} b_{21})' A_{21} (z_2 - A_{21}^{-1} b_{21}) \\
&\quad - b_{21}' A_{21}^{-1} b_{21} + c_{21}. \qquad (42)
\end{aligned}
$$

Using the following notation

$$d_{21} \stackrel{\Delta}{=} -b_{21}' A_{21}^{-1} b_{21} + c_{21} \qquad (43)$$

$$G_{21} \stackrel{\Delta}{=} (z_2 - A_{21}^{-1} b_{21})' A_{21} (z_2 - A_{21}^{-1} b_{21}) \qquad (44)$$

expression (34) becomes

$$D_{21} - D_{22} = G_{21} + d_{21}. \qquad (45)$$

Note that (44) can be rewritten as

$$G_{21} \stackrel{\Delta}{=} (z_2 - \hat{z}_2 + \hat{z}_2 - A_{21}^{-1} b_{21})' A_{21} (z_2 - \hat{z}_2 + \hat{z}_2 - A_{21}^{-1} b_{21}) \qquad (46)$$

which would be exactly noncentral chi-square distributed if the matrix in the quadratic form would have been the covariance of z_2. As in (28), define

$$G_{21}^* \stackrel{\Delta}{=} \beta_{21} G_{21} \qquad (47)$$

where

$$\beta_{21} \stackrel{\Delta}{=} \frac{n_z}{\mathrm{tr}(A_{21} S_2)}. \qquad (48)$$

Then, G_{21}^* is approximately (by moment matching) noncentral chi-square distributed with n_z d.o.f. and noncentrality parameter

$$\lambda_{21}^* \stackrel{\Delta}{=} \beta_{21} (\hat{z}_2 - A_{21}^{-1} b_{21})' A_{21} (\hat{z}_2 - A_{21}^{-1} b_{21}). \qquad (49)$$

This is written as

$$G_{21}^* \sim \chi^2_{n_z, \lambda_{21}^*}. \qquad (50)$$

A similar definition yields G_{12}^*, which is the negative of the r.h.s. of (33). The pdf of G_{12}^* is the same as in (50) with the indices 1 and 2 switched. Furthermore, G_{21}^* and G_{12}^* are independent.

The misassociation event for a Global assignment between tracks 1 and 2 is thus

$$\mathrm{MA}^{\mathrm{G}}_{21,12} = \left\{ \frac{G_{21}^*}{\beta_{21}} + d_{21} < - \left[\frac{G_{12}^*}{\beta_{12}} + d_{12} \right] \right\}. \qquad (51)$$

The probability of the above is then obtained as

$$\mathsf{P}_{\mathrm{MA}^{\mathrm{G}}_{21,12}} = \int_0^\infty X^2_{n_z, \lambda_{21}^*} \left[-\beta_{21} \left(\frac{x}{\beta_{12}} + d_{12} + d_{21} \right) \right] \chi^2_{n_z, \lambda_{12}^*}(x) dx. \qquad (52)$$

Note from (43) that $d_{12} = -d_{21}$ and thus (52) becomes

$$\mathsf{P}_{\mathrm{MA}^{\mathrm{G}}_{21,12}} = \int_0^\infty X^2_{n_z, \lambda_{21}^*} \left[-\frac{\beta_{21}}{\beta_{12}} x \right] \chi^2_{n_z, \lambda_{12}^*}(x) dx. \qquad (53)$$

For the second approach, the mean and variance of the differences[7] $\Delta_2 \stackrel{\Delta}{=} D_{21} - D_{22}$ and $\Delta_1 \stackrel{\Delta}{=} D_{11} - D_{12}$ are required to approximate their distributions by a Gaussian pdfs. From (42) we have that $D_{21} - D_{22}$ is a quadratic expression in z_2, and similarly for $D_{11} - D_{12}$, thus

$$\Delta_2 = D_{21} - D_{22} = z_2' A_{21} z_2 - 2 b_{21} z_2 + c_{21} \qquad (54)$$

$$\Delta_1 = D_{11} - D_{12} = z_1' A_{21} z_1 - 2 b_{21} z_1 + c_{21}. \qquad (55)$$

Note that the above two RVs are independent. Using the results in the appendix showing the variance of a quadratic form, one has (approximately)

$$\Delta_i \sim \mathcal{N}(\mu_i, \sigma_i^2) \qquad (56)$$

where

$$\mu_i = \mathrm{tr}(A_{21} S_i) + \hat{z}_i' A_{21} \hat{z}_i - 2 b_{21}' \hat{z}_i + c_{21} \qquad (57)$$

$$\sigma_i^2 = 2\mathrm{tr}(A_{21} S_i A_{21} S_i) + 4[A_{21} \hat{z}_i - b_{21}]' S_i [A_{21} \hat{z}_i - b_{21}]. \qquad (58)$$

As in Section 4.1, these two Gaussian random variables are independent, so the misassociation probability can be calculated in terms of the cumulative density function Φ of a standard Normal random variable as

$$
\begin{aligned}
\mathsf{P}_{\mathrm{MA}^{\mathrm{G}}_{21,12}} &= \mathsf{P}\{\Delta_2 - \Delta_1 < 0\} \\
&= \mathsf{P}\left\{ \frac{\Delta_2 - \Delta_1 - \mu_2 + \mu_1}{(\sigma_1^2 + \sigma_2^2)^{1/2}} < \frac{\mu_1 - \mu_2}{(\sigma_1^2 + \sigma_2^2)^{1/2}} \right\} \\
&= \Phi\left(\frac{\mu_1 - \mu_2}{(\sigma_1^2 + \sigma_2^2)^{1/2}} \right). \qquad (59)
\end{aligned}
$$

4.2.2. Global assignment misassociation with unequal innovation covariances—exact solution

The approximation methods of the previous subsection will be shown to work well when the covariances are not very different, that is, when the matrix $A_{21} = S_1^{-1} - S_2^{-1}$ has small eigenvalues compared to the values of the element in b_{21}. If this is not the case, something that happens when S_1 greatly differs from S_2, the distribution of the quadratic form is not easy to approximate, and a method to obtain the true distribution is required.

In the following subsection a method to numerically obtain the cdf of a noncentral quadratic function will be delineated, following [8]. Analytical expressions have been derived via a series representation, similarly to [12], for the case of real Gaussian random variables, but the convergence of such series is not assured, and hence only the numerical integration method is presented.

[7]Different notations are used than in Section 4.1 because their expressions are different.

The quadratic function of Gaussian random variables (42) is replicated here without subscripts for clarity

$$Q(Z) = Z'AZ + Z'b + b'Z + c$$
$$= (Z + A^{-1}b)'A(Z + A^{-1}b) - b'A^{-1}b + c \tag{60}$$

where the N-vector Z is Gaussian

$$Z \sim \mathcal{N}(\mu, \Sigma). \tag{61}$$

Neglecting the constant, (60) can be expressed as a weighted sum of noncentral chi-square random variables by writing Z in terms of $X \sim \mathcal{N}(0, I)$

$$Z = \Sigma^{1/2}X + \mu \tag{62}$$

as

$$Q(Z) = [X + \Sigma^{-1/2}(\mu + A^{-1}b)]'[\Sigma^{1/2}]'$$
$$\times A\Sigma^{1/2}[X + \Sigma^{-1/2}(\mu + A^{-1}b)]$$
$$= \sum_{k=1}^{N} \lambda_k \chi^2_{1,\rho_k^2} \tag{63}$$

where the matrix $[\Sigma^{1/2}]'A\Sigma^{1/2}$ has distinct eigenvalues λ_k and eigenvector matrix T, and

$$\rho \stackrel{\Delta}{=} [\rho_1 \ldots \rho_N]' = T\Sigma^{-1/2}(\mu + A^{-1}b). \tag{64}$$

The characteristic function of $Q(Z)$ is

$$\phi(t) = \prod_{k=1}^{N}(1 - 2i\lambda_k t)^{-1/2} \exp\left(i \sum_{r=1}^{N} \frac{\rho_r^2 \lambda_r t}{1 - 2i\lambda_r t}\right). \tag{65}$$

This function can be inverted as in [8], yielding

$$P(Q < q) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin\theta(u)}{u\kappa(u)} du \tag{66}$$

where

$$\theta(u) = \frac{1}{2}\sum_{k=1}^{N}\left[\tan^{-1}(\lambda_k u) + \frac{\rho_k^2 \lambda_k u}{1 + \lambda_k^2 u^2}\right] - \frac{qu}{2} \tag{67}$$

$$\kappa(u) = \prod_{k=1}^{N}(1 + \lambda_k^2 u^2)^{1/4} \exp\left(\frac{1}{2}\sum_{r=1}^{N} \frac{\rho_r^2 \lambda_r^2 u^2}{1 + \lambda_r^2 u^2}\right). \tag{68}$$

Also the probability density function is obtained as

$$f(q) = \frac{1}{\pi}\int_0^\infty \frac{\cos\theta(u)}{\kappa(u)} du. \tag{69}$$

These integrals can be truncated to some finite upper limit since $\kappa(u)$ is an increasing function in u, and the integral is approximated using Simpson's rule, following the suggestions in [8].

Having both the pdf and cdf of the quadratic functions, the exact probability of misassociation is obtained as in the previous sections. While this characteristic function based procedure is exact, unlike the moment matching procedure from the previous subsection, it is more costly. However, the moment matching procedure is accurate enough in certain circumstances, to be specified in the next section.

5. FORMULATION OF THE T2T ASSOCIATION PROBLEM

In this section we consider the case of two sensors m, n, generating track estimates from two targets i, j. The track estimate of target i generated by sensor m after receiving the measurement at time k is noted as $\hat{x}_i^m(k \mid k)$. Some authors [6, 13] have tried to obtain the T2T misassociation probability, but without considering the correlation of the estimation errors (due to the common process noise [1]) and have also considered only a single track, thus no global association results have been reported.

In the case of Gaussian measurements, the log likelihood ratio for the common origin association consists of two terms. One is the normalized distance squared (NDS) and the other is a ratio of the determinant of the innovation covariance matrix and the density μ_{ex} of extraneous tracks [4]. In the case of tracks $\hat{x}_j^n(k \mid k)$ and $\hat{x}_i^m(k \mid k)$ the NDS takes the form

$$D_{ij}^{mn}(k) = (\hat{x}_i^m(k \mid k) - \hat{x}_j^n(k \mid k))'[T_{ij}^{mn}(k)]^{-1}(\hat{x}_i^m(k \mid k) - \hat{x}_j^n(k \mid k)) \tag{70}$$

where $T_{ij}^{mn}(k)$ is the track difference covariance, which will be given later.

The likelihood based association cost for tracks $\hat{x}_j^n$ and $\hat{x}_i^m$ is the negative log likelihood ratio (NLLR)

$$C_{ij}^{mn} = D_{ij}^{mn} + \ln(|2\pi T_{ij}^{mn}|^{1/2}/\mu_{ex}). \tag{71}$$

The superscripts will be dropped when possible as we are checking for association between tracks originated from sensors m and n only and not across time. Also the time index k will be dropped, for brevity, thus for example $T_{ij}^{mn}(k)$ becomes T_{ij}.

5.1. Nearest Neighbor T2T Association Criterion

Using the NLLR cost as a modified distance definition, the Nearest Neighbor (NN) misassociation event ($\mathrm{MA}_{ij}^{\mathrm{NN}}$) that the estimate of target i obtained by sensor m is assigned to track j from sensor n instead of being assigned to track i from sensor n, is defined by

$$\{\mathrm{MA}_{21}^{\mathrm{NN}}\} \stackrel{\Delta}{=} \{C_{ij} < C_{ii}\}. \tag{72}$$

Consider each of these cost terms separately, and note that each estimate can be expressed as its true value plus the error term $\hat{x}_i^m(k \mid k) = x_i(k) + \tilde{x}_i^m(k \mid k)$, where $x_i(k)$ is the true state of target i (regardless of the sensor).

For the term C_{ii}, i.e., the cost of associating the tracks corresponding to target i obtained at the two sensors, the covariance matrix

$$\begin{aligned}T_{ii} &= E\{(\tilde{x}_i^m - \tilde{x}_i^n)'(\tilde{x}_i^m - \tilde{x}_i^n)\}\\ &= E\{(\tilde{x}_i^m)'\tilde{x}_i^m\} + E\{(\tilde{x}_i^n)'\tilde{x}_i^n\}\\ &\quad - E\{(\tilde{x}_i^m)'\tilde{x}_i^n\} - E\{(\tilde{x}_i^n)'\tilde{x}_i^m\}\\ &= P_i^m + P_i^n - P_i^{mn} - P_i^{nm} \end{aligned} \tag{73}$$

is required.

The autocovariance terms are obtained from the estimation algorithm (a Kalman filter in our case) and the crosscovariance is present due to the common process noise. This crosscovariance terms can be obtained in an iterative way [1] by the Lyapunov type equation

$$\begin{aligned}P_i^{mn}(k) &= E\{(\tilde{x}_i^m)'\tilde{x}_i^n\}\\ &= [I - W_i^m H_i^m][F P_i^{mn}(k-1)F' + Q][I - W_i^n H_i^n]'.\end{aligned} \tag{74}$$

The distance term in the cost can then be written as

$$\begin{aligned}D_{ii} &= (x_i + \tilde{x}_i^m - x_i - \tilde{x}_i^n)'[T_{ii}(k)]^{-1}(x_i + \tilde{x}_i^m - x_i - \tilde{x}_i^n)\\ &= (\tilde{x}_i^m - \tilde{x}_i^n)'[T_{ii}(k)]^{-1}(\tilde{x}_i^m - \tilde{x}_i^n).\end{aligned} \tag{75}$$

This random variable has χ^2 distribution, but its dependence on the other distance term D_{ij} (through $\tilde{x}_i^m$) precludes the usage of the results obtained for the M2T association.

The term C_{ij} is the cost of associating track i obtained from sensor m to the track j obtained by sensor n. In this case the covariance matrix T_{ij} is simply $P_i^m + P_j^n$, as the track errors are not correlated. Then, the distance D_{ij} can be written as

$$\begin{aligned}D_{ij} &= (x_j + \tilde{x}_j^m - x_i - \tilde{x}_i^n)'[T_{ij}(k)]^{-1}(x_j + \tilde{x}_j^m - x_i - \tilde{x}_i^n)\\ &= (\tilde{x}_i^m - \tilde{x}_j^n - c)'[T_{ij}(k)]^{-1}(\tilde{x}_i^m - \tilde{x}_j^n - c)\end{aligned} \tag{76}$$

where $c = x_i - x_j$ is the separation between tracks.

The exact probability of misassociation can be obtained as

$$\begin{aligned}\mathsf{P}_{\mathrm{MA}_{ij}^{\mathrm{NN}}} &= \mathsf{P}\{C_{ij} < C_{ii}\}\\ &= \mathsf{P}\{D_{ij} - D_{ii} + \gamma_{ij} < 0\}\\ &= \int_{\mathcal{A}} [(\tilde{x}_i^m - \tilde{x}_j^n - c)'[T_{ij}(k)]^{-1}(\tilde{x}_i^m - \tilde{x}_j^n - c)\\ &\qquad - (\tilde{x}_i^m - \tilde{x}_i^n)'[T_{ii}(k)]^{-1}(\tilde{x}_i^m - \tilde{x}_i^n) + \gamma_{ij}]\\ &\qquad \times p(\tilde{x}_i^m, \tilde{x}_i^n, \tilde{x}_j^m) d\tilde{x}_i^m d\tilde{x}_i^n d\tilde{x}_j^m\end{aligned} \tag{77}$$

where

$$\mathcal{A} = \{\{\tilde{x}_i^n \tilde{x}_i^m \tilde{x}_j^n\} : (D_{ij} - D_{ii}) < 0\} \tag{78}$$

and

$$\begin{aligned}\gamma_{ij} &= \ln(|2\pi T_{ij}^{mn}|^{1/2}/\mu_{ex}) - \ln(|2\pi T_{ij}^{mn}|^{1/2}/\mu_{ex})\\ &= \ln(|T_{ij}^{mn}|^{1/2}/|T_{ii}^{mn}|^{1/2}).\end{aligned} \tag{79}$$

The integration region $\mathcal{A}$ is very difficult to find, and this does preclude the usage of (77) for the calculation of the misassociation probability. Another approach is to obtain the pdf of the cost difference, but the fact that two of the estimates are correlated makes the exact pdf calculation very complex. These are the reasons that lead to the moment matching approach technique described next.

As the distance formulas involve quadratic terms, closed form first and second order moments can be obtained, which depend only on the correlation matrices involved. So, if we define

$$\eta_1 = D_{ij} - D_{ii} + \gamma_{ij} \tag{80}$$

the first and second moments of η_1, μ_{η_1} and $\sigma_{\eta_1}^2$, are obtained using (133) and (134) in the appendix. These moments are used to match both a Gaussian distribution as well as a shifted chi-square distribution, to obtain approximate misassociation probabilities.

The Gaussian approximation $\xi_1 \approx \eta_1$ follows by defining

$$\xi_1 \sim \mathcal{N}(\mu_{\eta_1}, \sigma_{\eta_1}^2). \tag{81}$$

Then the approximate misassociation probability is given by

$$\mathsf{P}_{\mathrm{MA}_{ij}^{\mathrm{NN}}}^{\mathrm{G}} = \Phi\left(\frac{\mu_{\eta_1}}{\sigma_{\eta_1}}\right) \tag{82}$$

where $\Phi(\cdot)$ is the normal standard cumulative distribution function.

The chi-square approximation $\zeta_1 \approx \eta_1$ is based on the definition of the random variable ζ_1 in terms of a shifted chi-square random variable w with k degrees of freedom

$$\zeta_1 = \varrho_1 + w \tag{83}$$

where

$$\varrho_1 = \mu_{\eta_1} - \sigma_{\eta_1}^2/2 \tag{84}$$

$$k = \sigma_{\eta_1}^2/2 \tag{85}$$

and $X_k^2(\cdot)$ is the cumulative distribution function for a chi-square random variable with k degrees of freedom. Then the approximate misassociation probability is given by

$$\mathsf{P}_{\mathrm{MA}_{ij}^{\mathrm{NN}}}^{\chi^2} = X_k^2(\varrho_1). \tag{86}$$

5.2. Global T2T Association Criterion

Similarly to the NN case, the misassociation event $\mathrm{MA}_{ij}^{\mathrm{G}}$ that the estimate of target i and j obtained by sensor m are respectively assigned using a global approach to tracks j and i from sensor n instead of being assigned to tracks i and j from sensor n, is defined by

$$\{\mathrm{MA}^{\mathrm{G}}\} \stackrel{\Delta}{=} \{C_{ij} + C_{ji} < C_{ii} + C_{jj}\}. \tag{87}$$

The exact probability of misassociation is again very difficult to obtain, thus a moment matching approach to a Gaussian and a shifted chi-square random variable will be used to obtain two approximations. Defining

$$\eta_2 = D_{ij} + D_{ji} - D_{ii} - D_{jj} + \gamma \tag{88}$$

where

$$\gamma = \ln(|T_{ij}^{mn}|^{1/2}|T_{ji}^{mn}|^{1/2}/(|T_{ii}^{mn}|^{1/2}|T_{jj}^{mn}|^{1/2})). \tag{89}$$

The first and second moments of η_2, μ_{η_2} and $\sigma_{\eta_2}^2$, are obtained using (140) and (141) in the appendix.

The Gaussian approximation $\xi_2 \approx \eta_2$ is, as before,

$$\xi_2 \sim \mathcal{N}(\mu_{\eta_2}, \sigma_{\eta_2}^2) \tag{90}$$

and this yields the approximate misassociation probability as

$$\mathsf{P}_{\mathrm{MA}^{\mathrm{G}}}^{\mathcal{N}} = \Phi\left(\frac{\mu_{\eta_2}}{\sigma_{\eta_2}}\right). \tag{91}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function.

The chi-square approximation $\zeta_2 \approx \eta_2$ is also as before based on the definition of the random variable ζ_2 in terms of a shifted chi-square random variable ς with k degrees of freedom

$$\zeta_2 = \varrho_2 + \varsigma \tag{92}$$

where

$$\varrho_2 = \mu_{\eta_2} - \sigma_{\eta_2}^2/2 \tag{93}$$

$$k = \sigma_{\eta_2}^2/2. \tag{94}$$

Denoting $X_k^2(\cdot)$ to the cumulative distribution function of a chi-square random variable with k degrees of freedom, the approximate misassociation probability is given by

$$\mathsf{P}_{\mathrm{MA}^{\mathrm{G}}}^{\chi^2} = X_k^2(\varrho_2). \tag{95}$$

6. SIMULATION RESULTS

A number of cases with unequal covariances are considered to compare the techniques for the Nearest Neighbor of Section 3 to Monte Carlo results. As a limiting case, the equal covariance situation is also considered.

Two targets, moving in 3 dimensions, are considered. Their motion is modeled by a NCV (nearly constant velocity) model [3] in each Cartesian coordinate with Gaussian zero-mean white process noise with PSD (power spectral density) $\tilde{q}$, uncorrelated across the coordinates. Position measurements z_s are obtained with probability of detection one at sampling intervals of T in spherical ("s") coordinates (range, azimuth and elevation) with additive Gaussian zero-mean white noise with covariance

$$R_s = \mathrm{diag}[\sigma_r^2,\ \sigma_a^2,\ \sigma_e^2]. \tag{96}$$

These measurements are converted into Cartesian coordinates ("C") in the standard manner, resulting in z_C, with covariance matrix (at a particular position x_p relative to the radar), denoted as $R_C(x_p)$. The tracking filter is then linear [3]. No clutter or false measurements are considered in this work. In an actual tracking algorithm the covariance of the converted measurements is evaluated at the predicted position or at the measurement itself, whichever is more accurate. Here the converted measurement covariances (for the two targets) will be evaluated at the predicted locations of the corresponding measurements, $\hat{z}_{C_t}$, $t = 1,2$, which will be also a parameter of the evaluation to be carried out. These predicted measurements will quantify the separation between the targets.

To simulate a case where the measurement covariances are unequal, it is assumed that target t was observed n_t times, $t = 1,2$. These will yield the innovation covariances S_t, $t = 1,2$ (in Cartesian coordinates, denoted only with the target subscript for simplicity). With the values of $\hat{z}_{C_t}$, and S_t, $t = 1,2$, a random number generator will be used to generate the measurements

$$z_t \sim \mathcal{N}(\hat{z}_{C_t}, S_t), \qquad t = 1,2. \tag{97}$$

Let $z_t(j)$ denote the measurements in Monte Carlo run j, $j = 1,\ldots,N$. Using these, denote the indicator variable of the misassociation event (9) as

$$\chi(j) = \begin{cases} 1 & \text{if} \quad D(z_2,\hat{z}_1) < D(z_1,\hat{z}_1) \\ 0 & \text{otherwise} \end{cases} \tag{98}$$

where the distances $D(z,\hat{z})$ are defined in (12). The theoretical probability that the above indicator will be unity (i.e., a misassociation occurs) is given by (31), to be denoted now as P.

Thus the test whether the theoretical probability matches the outcomes (98) will be based on the statistic

$$\hat{\mathsf{P}} = \frac{1}{N}\sum_{j=1}^{N}\chi(j) \tag{99}$$

and, in order for P to be acceptable,[8] one has to have (see, e.g., [3], Sect. 2.6.4)

$$|\hat{\mathsf{P}} - \mathsf{P}| < 2\sqrt{\frac{\mathsf{P}(1-\mathsf{P})}{N}} \tag{100}$$

or,

$$|\hat{\mathsf{P}} - \mathsf{P}| < 2\sqrt{\frac{\hat{\mathsf{P}}(1-\hat{\mathsf{P}})}{N}} \tag{101}$$

based on the 95% probability region.

[8]Both $\hat{\mathsf{P}}$ and P will be subscripted later by NN for the Nearest Neighbor assignment misassociation and by G for the Global assignment.

TABLE I
The M2T Misassociation Probabilities for Various Covariances and Separations (Scenario 1) for the Nearest Neighbor Method

n_1	n_2	$S_1/10^4$			$S_2/10^4$			c	δ	$P_{MA^{NN}}$	$\hat{P}_{MA^{NN}}$
30	30	1.202	−1.182	−0.011	1.202	−1.182	−0.012	.03	0.306	.490	.495
		−1.182	1.202	−0.013	−1.182	1.202	−0.011	.1	1.019	.400	.400
		−0.011	−0.013	2.385	−0.012	−0.011	2.385	.3	3.058	.063	.065
30	10	1.202	−1.182	−0.011	1.372	−1.352	−0.039	.03	0.305	.450	.455
		−1.182	1.202	−0.013	−1.352	1.374	−0.012	.1	1.017	.372	.369
		−0.011	−0.013	2.385	−0.039	−0.012	2.815	.3	3.051	.065	.058
30	5	1.202	−1.182	−0.011	1.810	−1.794	−0.085	.03	0.267	.322	.328
		−1.182	1.202	−0.013	−1.794	1.845	−0.062	.1	0.892	.275	.273
		−0.011	−0.013	2.385	−0.085	−0.062	4.164	.3	2.675	.069	.066

Note: The minor differences between the covariances of the two targets for $n_1 = n_2 = 30$ are due to the fact that the conversions from spherical to Cartesian coordinates of the measurement noise covariances (which amount to linearizations) were done at slightly different points, (109) and (110), respectively.

The following values for the parameters of the problem were considered:

$$\tilde{q} = 5 \text{ m}^2/\text{s}^3 \tag{102}$$

$$T = 1 \text{ s} \tag{103}$$

$$\sigma_r = 10 \text{ m} \tag{104}$$

$$\sigma_a = \sigma_e = 1 \text{ mrad} \tag{105}$$

$$x_p = [10^5 \;\; 10^5 \;\; 10^3]' \text{ m} \tag{106}$$

$$n_1 = 30 \tag{107}$$

$$n_2 = 5; 10; 30 \tag{108}$$

$$\hat{z}_1 = x_p \tag{109}$$

$$\hat{z}_2 = x_p + c[10^2 \;\; 10^2 \;\; 10^2]' \text{ m}. \tag{110}$$

In the above c is a coefficient that yields several separations.

In Scenario 1 the same radar located at the origin of the coordinate system is tracking both targets. In Scenario 2, target 2 has been tracked (for n_2 samples) by another radar located at $[2 \cdot 10^5, 0, 0]$. In this case the ellipsoids corresponding to the two covariance matrices are approximately perpendicular.

A "normalized separation distance" between the two targets, denoted as δ, is evaluated, following (16), according to

$$\delta^2 \triangleq (\hat{z}_2 - \hat{z}_1)'[S_1^{-1/2}]'S_2^{-1/2}(\hat{z}_2 - \hat{z}_1). \tag{111}$$

This gives a measure of the closeness of the two targets for the case of different innovation covariances.

6.1. M2T Misassociation with Nearest Neighbor Assignment

Table I shows for the values of n_i listed above, the resulting covariances S_i, and for the three separations, defined by $c = 0.03; 0.1; 0.3$, the resulting normalized separation distances, the theoretical $P_{MA^{NN}}$ (based on the non-central χ^2 distribution, presented in Section 4) as well as the average $\hat{P}_{MA^{NN}}$ from 1000 Monte Carlo runs. In all cases the differences between these probabilities is well within the limits given in (100) for $N = 1000$. Therefore the theoretical misassociation probabilities $P_{MA^{NN}}$, as computed by the method presented in Section 4, are remarkably accurate.

Note that for decreasing n_2 the $P_{MA^{NN}}$ for equivalent separation does also decrease. This seems counterintuitive, as larger number of measurements correspond to having more information available at the tracker, and thus a better (smaller) probability of misassociation is expected. The reason for this phenomenon is in the relative size of the innovation covariance matrices. When these are similar the chance of confusing the measurements is high as the measurements from both targets are within the same distance to any of tracks. Instead, if one of the ellipsoids is larger than the other, the distance of some of the measurements corresponding to this track (the ones occurring outside the smaller ellipsoid) to the center of the smaller covariance track will be larger, thus making it harder to misassociate them.

6.2. M2T Misassociation with Global Assignment

Table II shows the comparison of Nearest Neighbor association results with the approximate Global association misassignment probability evaluation algorithms from 4.2 (Gaussian fit and χ^2 fit via moment matching) with the results for Monte Carlo runs.

The χ^2 fit for the Global assignment approach does not work when the covariance matrices are very similar, as the noncentrality parameter depends on the inverse of the difference of these matrices. On the other hand, the Gaussian fit method does give accurate results over all the range of parameters, showing that for this Scenario the linear term dominates over the quadratic.

For the case of Scenario 2, the covariance ellipses S_1 and S_2 are quite different as a result of the location of the sensors during the initial estimation periods. In this case, prior to the current time when the association is considered, sensor 1 has been tracking target 1 only

TABLE II
The M2T Misassociation Probabilities for Various Covariances and Separations using Both the Global and Nearest Neighbor Approaches (Scenario 1)

n_1	n_2	c	$\mathsf{P}_{\mathrm{MA^{NN}}}$	$\hat{\mathsf{P}}_{\mathrm{MA^{NN}}}$	$\hat{\mathsf{P}}_{\mathrm{MA^{G}}}$	$\mathsf{P}^{\mathcal{N}}_{\mathrm{MA^{G}}}$	$\mathsf{P}^{\chi^2}_{\mathrm{MA^{G}}}$
		.03	.490	.495	.410	.416	N/A
30	30	.1	.400	.400	.235	.235	N/A
		.3	.063	.065	.016	.015	N/A
		.03	.450	.455	.401	.399	.126
30	10	.1	.372	.369	.230	.232	N/A
		.3	.065	.058	.016	.015	N/A
		.01	.322	.328	.316	.312	.312
30	5	.1	.275	.273	.224	.235	.231
		.3	.069	.066	.029	.039	.032

TABLE IV
The T2T Misassociation Probabilities for Different Track Accuracies and Separations using Both the Global and Nearest Neighbor Approaches

n_1	n_2	c	$\mathsf{P}^{\mathcal{N}}_{\mathrm{MA^{NN}}}$	$\mathsf{P}^{\chi^2}_{\mathrm{MA^{NN}}}$	$\hat{\mathsf{P}}_{\mathrm{MA^{NN}}}$	$\mathsf{P}^{\mathcal{N}}_{\mathrm{MA^{G}}}$	$\mathsf{P}^{\chi^2}_{\mathrm{MA^{G}}}$	$\hat{\mathsf{P}}_{\mathrm{MA^{G}}}$
		.01	.484	.553	.480	.464	.476	.464
10	10	.2	.236	.251	.225	.123	.112	.100
		.4	.043	.026	.028	.073	.011	.015
		.01	.345	.396	.321	.258	.269	.240
30	30	.2	.252	.274	.233	.162	.161	.149
		.4	.116	.100	.096	.055	.035	.034
		.01	.466	.542	.457	.344	.394	.339
10	50	.2	.344	.394	.339	.236	.244	.227
		.4	.165	.161	.152	.066	.049	.052

TABLE III
The M2T Misassociation Probabilities for Various Covariances and Separations using the Global and Nearest Neighbor Approaches (Scenario 2)

n_1	n_2	$S_1/10^4$			$S_2/10^4$			c	$\hat{\mathsf{P}}_{\mathrm{MA^{NN}}}$	$\hat{\mathsf{P}}_{\mathrm{MA^{G}}}$	$\mathsf{P}^{chf}_{\mathrm{MA^{G}}}$
		1.202	−1.182	−0.011	1.2027	−0.8072	−0.0126	.03	.169	.136	.134
30	30	−1.182	1.202	−0.013	−0.8072	1.2028	−0.0088	.1	.164	.130	.129
		−0.011	−0.013	2.385	−0.0126	−0.0088	2.3852	.3	.131	.103	.095
		1.202	−1.182	−0.011	1.3724	−0.6369	−0.0395	.03	.118	.101	.100
30	10	−1.182	1.202	−0.013	−0.6369	1.3740	−0.0317	.1	.116	.097	.096
		−0.011	−0.013	2.385	−0.0395	−0.0317	2.8157	.3	.108	.086	.081
		1.202	−1.182	−0.011	1.8106	−0.1957	−0.0853	.01	.070	.061	.062
30	5	−1.182	1.202	−0.013	−0.1957	1.8452	−0.0820	.1	.069	.060	.061
		−0.011	−0.013	2.385	−0.0853	−0.0820	4.1647	.3	.068	.059	.056

and sensor 2 has been tracking target 2 only because of occlusion conditions. Then, after the initial estimation periods, both targets are visible for sensor 1 and a centralized fusion architecture [1] is assumed. The two measurements of sensor 1 are to be associated with the two tracks—one from sensor 1, the other from sensor 2.

Because of the different past "history" of these tracks, the difference matrix A_{21} is no longer "close" to zero, and in general it can have positive and negative eigenvalues,[9] so the approximate algorithms for the Global assignment from Section 4.2 give inaccurate values (for the Nearest Neighbor the evaluation algorithm from Section 4 works well). Since the distribution of such quadratic form is difficult to find, the characteristic function based method of Section 4.2 is needed. The results are shown in Table III.

6.3. T2T Misassociation Probabilities

Consider again Scenario 2, where two sensors (local trackers) generate track estimates from two targets, but now the estimation is done simultaneously by the sensors, and the results transmitted to a fusion center. The probability of misassocition is again parameterized by the separation between the targets. In this case we are not interested in the measurement-to-track association (which is assumed to be done perfectly at each sensor) but in the track-to-track association performed at the fusion center. We consider that the local tracks are based on different numbers of measurements, n_1 for the first sensor and n_2 for the second, modeling different times of target acquisition. The probability of detection is considered to be 1.

Table IV shows the results obtained by the two approximation methods for the cases of NN and global association criteria. Also the misassociation probability estimated from 1000 Monte Carlo runs is shown to validate the results obtained.

It can be seen that for large target separation the probability of misassociation goes to zero, as expected, and that in this case the chi-square approximation is very accurate, and much better than the normal approximation. For smaller separation, when the misassociation probability is close to one half, the Gaussian approximation is better, although it may mismatch the real value by up to a 10%. Overall both approximations provide larger probabilities than the true one, so the lowest value estimate should be used to guarantee an error below 10%.

[9] This scenario was devised to see if the evaluation technique works for indefinite difference matrices.

The advantage of using global assignment vs. nearest neighbor is clear, unless the targets are so far apart that it is obvious how to do the association, or so close that no matter which method is used, the misassociation probability is around 0.5.

7. CONCLUSIONS

For the M2T association problem, the probability that the measurement from an extraneous target will be (mis)associated with a target of interest by the (local) Nearest Neighbor association was evaluated exactly for the case of equal track prediction covariances and approximately for the case of unequal covariances. It was shown that this misassociation probability depends on a particular—covariance weighted—norm of the difference between the targets' predicted measurements designated as the "separation" of the tracks. Numerical simulations confirm the accuracy of the solutions presented for the misassociation probabilities.

For the Global association, in the case of very dissimilar track covariances the approximation methods do not work, and a characteristic function based method, which is more expensive computationally but exact, was presented with excellent results. The probability formulas derived as well as the Monte Carlo runs show the benefit of the Global (G) vs. Nearest Neighbor (NN) associations, especially in the case of similar track covariances. Future work will involve considering the multiframe or multidimensional association (MDA) case.

The T2T association problem is harder, and only approximate results are presented, which nonetheless provide accurate results for both global and NN association criteria. The estimated probabilities are never off by more than a 10% of the true value. It has been shown that a chi-square matching of the statistic gives the best results when the separation is large, and for the case of smaller separations a Gaussian matching provides better results.

APPENDIX A. MOMENT MATCHING OF QUADRATIC FUNCTIONS: UNCORRELATED CASE

Consider the random variable defined by

$$v = u'Au + b'u + c \tag{112}$$

where A is any real $n \times n$ matrix, b is a real $n \times 1$ vector, c is a real scalar and u is a Gaussian random vector

$$u \sim \mathcal{N}(\mu, \Sigma). \tag{113}$$

Define the zero mean version of u as $\tilde{u} = u - \mu$, an rewrite

$$\begin{aligned} v &= (\tilde{u} + \mu)'A(\tilde{u} + \mu) + b'(\tilde{u} + \mu) + c \\ &= \tilde{u}'A\tilde{u} + \tilde{u}'A\mu + \mu'A\tilde{u} + \mu'A\mu + b'\tilde{u} + b'\mu + c \\ &= \tilde{u}'A\tilde{u} + \tilde{b}'\tilde{u} + \tilde{c} \end{aligned} \tag{114}$$

where $\tilde{b} = A'\mu + A\mu + b$ and $\tilde{c} = \mu A'\mu + b'\mu + c$ The mean value of v is

$$\begin{aligned} E\{v\} = \bar{v} &= E\{\tilde{u}'A\tilde{u}\} + \tilde{b}'E\{\tilde{u}\} + \tilde{c} \\ &= \mathrm{tr}(A\Sigma) + \tilde{c} \end{aligned} \tag{115}$$

where the expected value of a quadratic form is taken from [3] Section 1.4.15 The variance of v is

$$\begin{aligned} \mathrm{Var}\{v\} &= E\{(v - \bar{v})^2\} = E\{(\tilde{u}'A\tilde{u} + \tilde{b}'\tilde{u} - \mathrm{tr}(A\Sigma))^2\} \\ &= E\{\tilde{u}'A\tilde{u}\tilde{u}'A\tilde{u}\} + \tilde{b}'E\{\tilde{u}\tilde{u}'\}\tilde{b} + \mathrm{tr}(A\Sigma)^2 + 2E\{\tilde{u}'A\tilde{u}'\tilde{u}'\}\tilde{b} \\ &\quad - 2E\{\tilde{u}'A\tilde{u}\}\mathrm{tr}(A\Sigma) - 2\tilde{b}'E\{\tilde{u}\}\mathrm{tr}(A\Sigma) \\ &= \mathrm{tr}(A\Sigma)^2 + 2\mathrm{tr}(A\Sigma A\Sigma) + \tilde{b}'\Sigma\tilde{b} + \mathrm{tr}(A\Sigma)^2 - 2\mathrm{tr}(A\Sigma)^2 \\ &= 2\mathrm{tr}(A\Sigma A\Sigma) + [(A + A')\mu + b]'\Sigma[(A + A')\mu + b] \end{aligned} \tag{116}$$

where, as before, a compact expression for the fourth moment of u is used, and the terms containing odd powers of u are zero.

APPENDIX B. MOMENT MATCHING OF QUADRATIC FUNCTIONS: CORRELATED CASE

Consider four random vectors $\hat{x}_i^m, \hat{x}_i^n, \hat{x}_j^m$ and $\hat{x}_j^n$, Gaussian distributed

$$\begin{bmatrix} \hat{x}_i^m \\ \hat{x}_i^n \\ \hat{x}_j^m \\ \hat{x}_j^n \end{bmatrix} = \begin{bmatrix} \tilde{x}_i^m + x_i \\ \tilde{x}_i^n + x_i \\ \tilde{x}_j^m + x_j \\ \tilde{x}_j^n + x_j \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} x_i \\ x_i \\ x_j \\ x_j \end{bmatrix}, \begin{bmatrix} P_i^m & P_{ii}^{mn} & 0 & 0 \\ P_{ii}^{nm} & P_i^n & 0 & 0 \\ 0 & 0 & P_j^m & P_{jj}^{mn} \\ 0 & 0 & P_{jj}^{nm} & P_j^n \end{bmatrix} \right)$$

the four possible distance terms are

$$D_{ij} = (\tilde{x}_i^m - \tilde{x}_j^n - s)'[P_i^m + P_j^n]^{-1}(\tilde{x}_i^m - \tilde{x}_j^n - s) \tag{117}$$

$$D_{ji} = (\tilde{x}_j^m - \tilde{x}_i^n + s)'[P_j^m + P_i^n]^{-1}(\tilde{x}_j^m - \tilde{x}_i^n + s) \tag{118}$$

$$D_{ii} = (\tilde{x}_i^m - \tilde{x}_i^n)'[P_i^m + P_i^n - P_{ii}^{mn} - P_{ii}^{nm}]^{-1}(\tilde{x}_i^m - \tilde{x}_i^n) \tag{119}$$

$$D_{jj} = (\tilde{x}_j^m - \tilde{x}_j^n)'[P_j^m + P_j^n - P_{jj}^{mn} - P_{jj}^{nm}]^{-1}(\tilde{x}_j^m - \tilde{x}_j^n) \tag{120}$$

where $s = x_j - x_i$.

For convenience, define

$$M = [P_i^m + P_j^n]^{-1}; \; N = [P_i^m + P_i^n - P_{ii}^{mn} - P_{ii}^{nm}]^{-1} \tag{121}$$

$$K = [P_j^m + P_i^n]^{-1}; \; Q = [P_j^m + P_j^n - P_{jj}^{mn} - P_{jj}^{nm}]^{-1}. \tag{122}$$

To obtain the first and second moments for the difference of distances, the following tools are needed.

From [3], the moments for quadratic and quartic zero mean Gaussian random vectors $x \sim \mathcal{N}(0, R_x)$ are given by

$$E\{x'W_1x\} = \mathrm{tr}(W_1R_x) \quad (123)$$

$$E\{x'W_1xx'W_2x\} = \mathrm{tr}(W_1R_x)\mathrm{tr}(W_2R_x) + 2\mathrm{tr}(W_1R_xW_2R_x). \quad (124)$$

If two zero mean Gaussian random vectors x and y are correlated through R_{xy}, the vector y can be written as

$$y = Rx + Tw \quad (125)$$

where $w \sim \mathcal{N}(0, I)$ and

$$R = R_{xy}'Rx^{-1} \quad (126)$$

$$T = (Ry - R_{xy}'Rx^{-1}R_{xy})^{1/2}. \quad (127)$$

In the above $\Xi^{1/2}$ denotes the Cholesky factor of Ξ, so that $\Xi^{1/2}(\Xi^{1/2})' = \Xi$.

For the NN association case we are interested in the moments of the difference $d_1 = D_{ij} - D_{ii}$. In this case the variables $\tilde{x}_i^m$ and $\tilde{x}_i^n$ are correlated, and can be related as

$$\tilde{x}_i^n = A\tilde{x}_i^m + Bw \quad (128)$$

where $w \sim \mathcal{N}(0, I)$ and

$$A = (P_{ii}^{mn})'(P_i^m)^{-1} \quad (129)$$

$$B = (P_i^n - (P_{ii}^{mn})'(P_i^m)^{-1}P_{ii}^{mn})^{1/2}. \quad (130)$$

Then we have

$$\mu_{d_1} = E\{(\tilde{x}_i^m - \tilde{x}_j^n - s)'M(\tilde{x}_i^m - \tilde{x}_j^n - s) - (\tilde{x}_i^m - \tilde{x}_i^n)'N(\tilde{x}_i^m - \tilde{x}_i^n)\} \quad (131)$$

$$= E\{(\tilde{x}_i^m)'(M - N)\tilde{x}_i^m - 2(\tilde{x}_i^m)'M(\tilde{x}_j^n + s) + (\tilde{x}_j^n)'M\tilde{x}_j^n + s'Ms - 2(\tilde{x}_j^n)'Ms + (\tilde{x}_i^m)'N\tilde{x}_i^n - (\tilde{x}_i^n)'N\tilde{x}_i^n\} \quad (132)$$

$$= \mathrm{tr}((M - N)P_i^m) + \mathrm{tr}(MP_j^n) + \mathrm{tr}(2NAP_i^m) - \mathrm{tr}(NP_i^n). \quad (133)$$

Define $\tilde{d}_1 = d_1 - s'Ms$, so that $\mathrm{cov}(d_1) = \mathrm{cov}(\tilde{d}_1)$ as s is a constant. Then after some algebraic operations

$$\begin{aligned}
E\{\tilde{d}_1^2\} &= E\{[(\tilde{x}_i^m - \tilde{x}_j^n - s)'M(\tilde{x}_i^m - \tilde{x}_j^n - s) - (\tilde{x}_i^m - \tilde{x}_i^n)'N(\tilde{x}_i^m - \tilde{x}_i^n) - s'Ms]^2\} \\
&= \mathrm{tr}((M - N)P_i^m)^2 + 2\mathrm{tr}((M - N)P_i^m(M - N)P_i^m) \\
&\quad + \mathrm{tr}(MP_j^n)^2 + 2\mathrm{tr}(MP_j^nMP_j^n) \\
&\quad - \mathrm{tr}(NP_i^n)^2 + 2\mathrm{tr}(NP_i^nNP_i^n) + 4\mathrm{tr}(Mss'MP_i^m) \\
&\quad + 4\mathrm{tr}(NAP_i^m)\mathrm{tr}(AN'P_i^m) + 8\mathrm{tr}(NAP_i^mAN'P_i^m) \\
&\quad + 4\mathrm{tr}(NBB'NP_i^m) + \mathrm{tr}((M - N)P_i^m)\mathrm{tr}(MP_j^n) \\
&\quad + 2\mathrm{tr}((M - N)P_i^m)\mathrm{tr}(NAP_i^m) + 4\mathrm{tr}((M - N)P_i^mNAP_i^m) \\
&\quad - \mathrm{tr}((M - N)P_i^m)\mathrm{tr}(ANAP_i^m) - \mathrm{tr}((M - N)P_i^mANAP_i^m) \\
&\quad - \mathrm{tr}((M - N)P_i^m)\mathrm{tr}((B'NB)m + 2\mathrm{tr}(MP_j^n)\mathrm{tr}((NAP_i^m) \\
&\quad - \mathrm{tr}(MP_j^n)\mathrm{tr}(NP_i^n) - 2\mathrm{tr}(NAP_i^m)\mathrm{tr}(A'NAP_i^m) \\
&\quad - 4\mathrm{tr}(NAP_i^mA'NAP_i^m) - 2\mathrm{tr}(NAP_i^m)\mathrm{tr}(B'NB) \\
&\quad - 4\mathrm{tr}(NBB'NAP_i^m).
\end{aligned} \quad (134)$$

For the global association case we are interested in the moments of

$$d_2 = D_{ij} + D_{ji} - D_{ii} - D_{jj}. \quad (135)$$

In this case the vectors $\tilde{x}_i^m$ and $\tilde{x}_i^n$ are correlated as before and (128) still holds. Also $\tilde{x}_j^m$ and $\tilde{x}_j^n$ are correlated and can be related as

$$\tilde{x}_j^n = C\tilde{x}_j^m + Dx \quad (136)$$

where $x \sim \mathcal{N}(0, I)$ and

$$C = (P_{jj}^{mn})'(P_j^m)^{-1} \quad (137)$$

$$D = \mathrm{chol}(P_j^n - (P_{jj}^{mn})'(P_j^m)^{-1}P_{jj}^{mn}). \quad (138)$$

Then, the moments of interest are

$$\begin{aligned}
\mu_{d_2} &= E\{(\tilde{x}_i^m - \tilde{x}_j^n - s)'M(\tilde{x}_i^m - \tilde{x}_j^n - s) \\
&\quad + (\tilde{x}_j^m - \tilde{x}_i^n - s)'K(\tilde{x}_j^m - \tilde{x}_i^n - s) \\
&\quad - (\tilde{x}_i^m - \tilde{x}_i^n)'N(\tilde{x}_i^m - \tilde{x}_i^n) - (\tilde{x}_j^m - \tilde{x}_j^n)'Q(\tilde{x}_j^m - \tilde{x}_j^n)\}
\end{aligned} \quad (139)$$

$$\begin{aligned}
&= \mathrm{tr}((M - N)P_i^m) + \mathrm{tr}((M - Q)P_j^n) \\
&\quad + \mathrm{tr}((K - N)P_i^n) + \mathrm{tr}((K - Q)P_j^m) \\
&\quad + 2\mathrm{tr}(NAP_i^m) + 2\mathrm{tr}(QCP_j^n).
\end{aligned} \quad (140)$$

Define $\tilde{d}_2 = d_2 - s'(M + k)s$, so that $\mathrm{cov}(d_2) = \mathrm{cov}(\tilde{d}_2)$ as s is a constant. Then after some algebraic operations

$$\begin{aligned}
E\{\tilde{d}_2^2\} &= \mathrm{tr}((M - N)P_i^m)^2 + 2\mathrm{tr}((M - N)P_i^m(M - N)P_i^m) \\
&\quad + \mathrm{tr}((M - Q)P_j^n)^2 + 2\mathrm{tr}((M - Q)P_j^n(M - Q)P_j^n) \\
&\quad + \mathrm{tr}((N - C)P_i^n)^2 + 2\mathrm{tr}((N - C)P_i^n(N - C)P_i^n) \\
&\quad + \mathrm{tr}((K - Q)P_j^m)^2 + 2\mathrm{tr}((K - Q)P_j^m(K - Q)P_j^m) \\
&\quad + \mathrm{tr}((-2M)P_j^n(-2M)'P_i^m) + \mathrm{tr}((-2M)ss'(-2M)'P_i^m) \\
&\quad + \mathrm{tr}((2M)ss'(2M)'P_j^n) + \mathrm{tr}((-2K)P_i^n(-2K)P_j^m) \\
&\quad + \mathrm{tr}((-2K)ss'(-2K)P_j^m) + \mathrm{tr}((-2K)ss'(-2K)P_i^n) \\
&\quad + \mathrm{tr}((2N)AP_i^m)\mathrm{tr}(A'(2N)AP_i^m) \\
&\quad + 2\mathrm{tr}((2N)AP_i^mA'(2N)AP_i^m) + \mathrm{tr}((2N)BB'(2N)'P_i^m) \\
&\quad + \mathrm{tr}(RCP_j^n)\mathrm{tr}(C'R'P_j^n) + 2\mathrm{tr}(RCP_j^nC'R'P_j^n) \\
&\quad + \mathrm{tr}(RDD'R'P_j^n) + \mathrm{tr}((M - N)P_i^m)\mathrm{tr}((M - Q)P_j^n) \\
&\quad + \mathrm{tr}((M - N)P_i^m)\mathrm{tr}(A'(N - C)AP_i^m) \\
&\quad + 2\mathrm{tr}((M - N)P_i^mA'(N - C)AP_i^m) \\
&\quad + \mathrm{tr}((M - N)P_i^m)\mathrm{tr}(B'(N - C)B) \\
&\quad + \mathrm{tr}((M - N)P_i^m)\mathrm{tr}((K - Q)P_j^m)
\end{aligned}$$

$$
\begin{aligned}
&+ \operatorname{tr}((M-N)P_i^m)\operatorname{tr}((2N)AP_i^m)\\
&+ 2\operatorname{tr}((M-N)P_i^m(2N)AP_i^m)\\
&+ \operatorname{tr}((M-N)P_i^m)\operatorname{tr}(R'CP_j^n)\\
&+ \operatorname{tr}((M-Q)P_j^n)\operatorname{tr}((N-C)P_i^n)\\
&+ \operatorname{tr}((M-Q)P_j^n)\operatorname{tr}(C'(K-Q)CP_j^n)\\
&+ 2\operatorname{tr}((M-Q)P_j^nC'(K-Q)CP_j^n)\\
&+ \operatorname{tr}((M-Q)P_j^n)\operatorname{tr}(D'(K-Q)D)\\
&+ \operatorname{tr}((M-Q)P_j^n)\operatorname{tr}((2N)AP_i^m)\\
&+ \operatorname{tr}((M-Q)P_j^n)\operatorname{tr}(R'CP_j^n)\\
&+ 2\operatorname{tr}((M-Q)P_j^nR'CP_j^n)\\
&+ \operatorname{tr}((N-C)P_i^n)\operatorname{tr}((K-Q)P_j^m)\\
&+ \operatorname{tr}(A'(N-C)AP_i^m)\operatorname{tr}((2N)AP_i^m)\\
&+ 2\operatorname{tr}(A'(N-C)AP_i^m(2N)AP_i^m)\\
&+ \operatorname{tr}(B'(N-C)B)\operatorname{tr}((2N)AP_i^m)\\
&+ 2\operatorname{tr}(A'(N-C)BB'(2N)'P_i^m)\\
&+ \operatorname{tr}((N-C)P_i^n)\operatorname{tr}(R'CP_j^n)\\
&+ \operatorname{tr}((K-Q)P_j^m)\operatorname{tr}((2N)AP_i^m)\\
&+ \operatorname{tr}(C'(K-Q)CP_j^n)\operatorname{tr}(R'CP_j^n)\\
&+ 2\operatorname{tr}(C'(K-Q)CP_j^nR'CP_j^n)\\
&+ \operatorname{tr}(D'(K-Q)D)\operatorname{tr}(R'CP_j^n)\\
&+ 2\operatorname{tr}(D'(K-Q)CP_j^nR'D)\\
&+ \operatorname{tr}((-2M)P_j^nC'(-2K)AP_i^m)\\
&+ \operatorname{tr}((-2M)ss'(-2K)'AP_i^m)\\
&- \operatorname{tr}((2M)ss'(-2K)CP_j^n)\\
&+ \operatorname{tr}((2N)AP_i^m)\operatorname{tr}(C'RP_j^n). \qquad (141)
\end{aligned}
$$

REFERENCES

[1] Y. Bar-Shalom and X. R. Li
Multitarget-Multisensor Tracking: Principles and Techniques.
YBS Publishing, 1995.

[2] Y. Bar-Shalom and W. D. Blair (Eds.)
Multitarget-Multisensor Tracking: Applications and Advances, vol. III.
Artech House, 2000.

[3] Y. Bar-Shalom, X. R. Li and T. Kirubarajan
Estimation with Applications to Tracking and Navigation: Algorithms and Software for Information Extraction.
J. Wiley and Sons, 2001.

[4] Y. Bar-Shalom, S. S. Blackman and R. J. Fitzgerald
The dimensionless score function for measurement to track association.
IEEE Transactions on Aerospace and Electronic Systems, **43**, 1 (Jan. 2007), 392–400.

[5] DCDFLIB (Double precision Cumulative Distribution Function LIBrary)
http://biostatistics.mdanderson.org/SoftwareDownload/SingleSoftware.aspx?Software_Id=21.

[6] A. Farina and R. Miglioli
Association of active and passive tracks for airborne sensors.
Signal Process. **69** ,3 (1998), 209–217.

[7] J. Ferry
XMAP: Track-to-track association with metric, feature, and target-type data.
In *Proceedings of the Conference on Information Fusion*, July 2006.

[8] J. P. Imhof
Computing the distribution of quadratic forms in normal variables.
Biometrika, **48** (1961), 419–426.

[9] M. Levedahl
An analysis of object designation performance using GNN and GNP correlation.
In *Proceedings SPIE Conference on Tracking Small Targets*, Orlando, FL, 2004, 441–451.

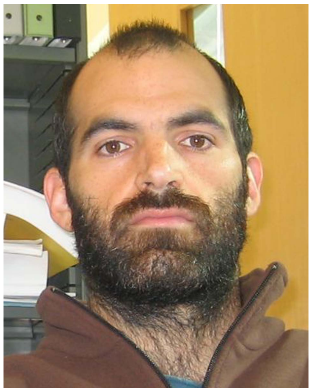
[10] S. Mori, K. C. Chang and C. Y. Chong
Performance analysis of optimal data association with application to multiple target tracking.
In Y. Bar-Shalom (Ed.), *Multitarget-Multisensor Tracking: Applications and Advances*, vol. II, Artech House, 1992 (reprinted by YBS Publishing, 1998).

[11] A. Poore and N. Rijavec
Lagrangian relaxation algorithm for multidimensional assignment problems arising from multitarget tracking.
SIAM Journal on Optimization **3**, 3 (Aug. 1993), 544–563.

[12] D. Raphaeli
Distribution of noncentral indefinite quadratic forms in complex normal variables.
IEEE Transactions on Information Theory, **42**, 3 (1996), 1002–1007.

[13] R. Saha
Analytical evaluation of an ESM/radar track association algorithm.
In *Proceedings SPIE Conference on Tracking Small Targets*, Orlando, FL, Apr. 1992, 338–347.

[14] L. Stone, M. Williams and T. Tran
Track-to-track association and bias removal.
In *Proceedings SPIE Conference on Tracking Small Targets*, Oct. 2001, 315–329. http://en.wikipedia.org/wiki/Noncentral_chi-square_distribution.



Javier Areta received the electrical engineer degree in 2001 from the University of La Plata, Argentina (UNLP). He is currently pursuing his Ph.D. in electrical and computer engineering at the University of Storrs, Connecticut. His main interests lie in the areas of tracking, data fusion and signal processing.

Yaakov Bar-Shalom (S'63—M'66—SM'80—F'84) was born on May 11, 1941. He received the B.S. and M.S. degrees from the Technion, Israel Institute of Technology, in 1963 and 1967 and the Ph.D. degree from Princeton University, Princeton, NJ, in 1970, all in electrical engineering.

From 1970 to 1976 he was with Systems Control, Inc., Palo Alto, CA. Currently he is Board of Trustees Distinguished Professor in the Dept. of Electrical and Computer Engineering and Marianne E. Klewin Professor in Engineering. He is also director of the ESP Lab (Estimation and Signal Processing) at the University of Connecticut. His research interests are in estimation theory and stochastic adaptive control and he has published over 360 papers and book chapters in these areas. In view of the causality principle between the given name of a person (in this case, "(he) will track," in the modern version of the original language of the Bible) and the profession of this person, his interests have focused on tracking.

He coauthored the monograph *Tracking and Data Association* (Academic Press, 1988), the graduate text *Estimation with Applications to Tracking and Navigation* (Wiley, 2001), the text *Multitarget-Multisensor Tracking: Principles and Techniques* (YBS Publishing, 1995), and edited the books *Multitarget-Multisensor Tracking: Applications and Advances* (Artech House, Vol. I 1990; Vol. II 1992, Vol. III 2000). He has been elected Fellow of IEEE for "contributions to the theory of stochastic systems and of multitarget tracking." He has been consulting to numerous companies, and originated the series of Multitarget Tracking and Multisensor Data Fusion short courses offered at Government Laboratories, private companies, and overseas.

During 1976 and 1977 he served as associate editor of the *IEEE Transactions on Automatic Control* and from 1978 to 1981 as associate editor of *Automatica*. He was program chairman of the 1982 American Control Conference, general chairman of the 1985 ACC, and cochairman of the 1989 IEEE International Conference on Control and Applications. During 1983–1987 he served as chairman of the Conference Activities Board of the IEEE Control Systems Society and during 1987–1989 was a member of the Board of Governors of the IEEE CSS. Currently he is a member of the Board of Directors of the International Society of Information Fusion and served as its Y2K and Y2K2 President. In 1987 he received the IEEE CSS distinguished Member Award. Since 1995 he is a distinguished lecturer of the IEEE AESS. He is corecipient of the M. Barry Carlton Awards for the best paper in the *IEEE Transactions on Aerospace and Electronic Systems* in 1995 and 2000, and received the 1998 University of Connecticut AAUP Excellence Award for Research, the 2002 J. Mignona Data Fusion Award from the DoD JDL Data Fusion Group, and the 2008 IEEE D. J. Picard Medal for Radar Technologies and Applications.

Ronald Rothrock was born on May 16, 1948. He received his B.S. degree in mechanical and aerospace engineering from Illinois Institute of Technology in 1970. He served six years in the U.S. Navy as an aviator, and earned his M.S. in operations research from Stanford University in 1977.

He worked for BDM International, Inc., (and its eventual purchaser, TRW) in McLean, VA, from 1977 to 1998, concentrating in the areas of defense communications and radar sensor networking and tracking. During 1998 he briefly worked for CSC and then joined SPARTA, Inc., in Arlington, VA, for which he continues to work as a chief engineer and director of a group that develops air vehicle, ballistic missile, and maritime surface vessel tracking, data conditioning, data association, classification, sensor registration, intelligent message pruning for peer-to-peer communications networks, and related algorithms, deliverable software, and simulations.